An Interesting Combinatorial Method in the Theory of Locally Finite Semigroups

T. C. Brown

Citation data: T.C. Brown, *An interesting combinatorial method in the theory of locally finite semigroups*, Pacific J. Math. **36** (1971), 285–289.

Abstract

Let X be a finite set, X^* the free semigroup (without identity) on X, let M be a finite semigroup, and let j be an epimorphism of X^* upon M. We give a simple proof of a combinatorial property of the triple (X, j, M), and exploit this property to get very simple proofs for these two theorems: 1. If j is an epimorphism of the semigroup S upon the locally finite semigroup T such that $j^{-1}(e)$ is a locally finite subsemigroup of S for each idempotent element e of T, then S is locally finite. 2. Throughout 1, replace "locally finite" by "locally nilpotent".

The method is simple enough, and yet powerful enough, to suggest its applicability in other contexts.

1 Introduction

Theorem 1 below was first proved by the author in [1] by a circuitous and laborious method. In the present paper it drops out easily from Lemma 2 below, as does Theorem 2, which is new. Lemma 2 was first discovered by J. Justin [3] as a generalization of Lemma 1, which is the author's [2]. The proof given here, however, is new, and is conceptually quite transparent, though apparently non-trivial. Justin has used Lemma 2 in an alternative proof of his generalization of Van der Waerden's theorem (on arithmetic

The symbol X^w denotes the set of sequences on X, regarded as "infinitely long words" in the alphabet X. If $x, y, z \in X^*$ and $s \in X^w$, then x, y, and z are each *factors* (x is a *left factor*) of the word xyz and of the sequence $xyzs$.

Let H be an infinite subset of X^*. We indicate now how to construct a sequence $S = a_1a_2\cdots \in X^w$ such that each left factor of s is a left factor of infinitely many of the words of H. Such a sequence is used several times in the proofs that follow, and is said to constructed *in the regular way* from H.

We choose the a_i's inductively. In view of the fact that H is infinite and X is finite, we choose a_1 to be an element of X which occurs infinitely often as the first letter in the words of H, and we denote by H_1 the (infinite) set of those words of H which have a_1 as the first letter. Thus a_1 is a left factor of infinitely many of the words of H. Now suppose that $a_1, \ldots, a_n \in X$ have been chosen so that $a_1a_2\cdots a_n$ is a left factor of each word in an infinite subset H_n of H. We choose a_{n+1} to be an element of X which occurs infinitely often as the $(n+1)$st letter in the words of H_n, and denote by H_{n+1} the (infinite) set of those words of H_n which have a_{n+1} as the $(n+1)$st letter. Thus $a_1a_2\cdots a_{n+1}$ is a left factor of infinitely many of the words of H.

3 Two Lemmas

Lemma 1. *Let $s = a_1a_2\cdots \in X^w$. Then there exist an element $x \in X$ and a fixed integer $\bar{k}$ such that for any n there are integers $i_1 < i_2 < \cdots < i_n$* (these depend on n) *with $x = a_{i_1} = a_{i_2} = \cdots = a_{i_n}$ and $i_{j+1} - i_j \leq \bar{k}$, $1 \leq j \leq n-1$.*

Proof. We proceed by induction on $|X|$, the cardinal of X. If $|X| = 1$, we are through. Assume the result for $|X| = k$, and suppose now that $|X| = k+1$, $X = \{x_1, \ldots, x_{k+1}\}$. Let $s = a_1a_2\cdots \in X^w$. If x_{k+1} is not missing from arbitrarily long factors of s we are done, hence we may assume that there is an infinite set H of factors of s from which x_{k+1} is missing. Thus $H \subset \{x_1, \ldots, x_k\}^*$, and we construct a sequence $t = b_1b_2\cdots \in \{x_1, \ldots, x_k\}^w$ from H in the regular way. By the induction hypothesis, there exist an element $x \in \{x_1, \ldots, x_k\}$ and an integer $\bar{k}$ such that for any n there are integers $i_1 < i_2 < \cdots < i_n$ with $x = b_{i_1} = b_{i_2} = \cdots = b_{i_n}$ and $i_{j+1} - i$

Let $\bar{s} \in \bar{X}^w$, and define the homomorphism $\bar{j}$ from $\bar{X}^*$ into T by setting $\bar{j}(\bar{w}) = j(w)$ for $\bar{w} \in \bar{X}^*$. Applying Lemma 2, we obtain an idempotent $e \in T$ and a fixed integer $\bar{k}$ such that for *any* n there are n consecutive factors $\bar{g}_1, \ldots, \bar{g}_n$ of $\bar{s}$ with $e = \bar{j}(\bar{g}_1) = \bar{j}(\bar{g}_2) = \cdots = \bar{j}(\bar{g}_n)$ and $|\bar{g}_j| \leq \bar{k}$, $1 \leq j \leq n$. By the definition of $\bar{j}$, we have $g_1, \ldots, g_n$ all in $j^{-1}(e)$, which is locally finite by assumption. Since $|\bar{g}_j| \leq \bar{k}$ there are only finitely many possibilities for the elements $g_1, \ldots, g_n$, hence the element $g_1 \cdots g_n$ always belongs to a certain fixed finite subsemigroup of $j^{-1}(e)$, no matter how large n is. Thus if n is taken sufficiently large, the factor $\bar{g}_1 \cdots \bar{g}_n$ of $\bar{s}$ will be contractible. Thus the sequence $\bar{s}$ is contractible. This completes the proof. □

Theorem 2. *Let j be an epimorphism of the semigroup S upon the locally nilpotent semigroup T such that $j^{-1}(e)$ is a locally nilpotent subsemigroup of S for each idempotent element e of T. Then S is locally nilpotent.*

Proof. The proof of Theorem 2 is practically the same as the proof of Theorem 1. Here instead of showing that every sequence $\bar{s}$ in $\bar{X}^w$ (same notation as in the proof of Theorem 1) has a contractible factor, one shows that every sequence $\bar{s}$ has a factor $\bar{w}$ with $w = 0$. □

References

[1] T.C. Brown, *On locally finite semigroups (in russian)*, Ukraine Math. J. **20** (1968), 732–738.

[2] ______, *On van der Waerden's theorem on arithmetic progressions*, Notices Amer. Math. Soc. **16** (1969), 245.

[3] J. Justin, *Généralisation du théorème de van der Waerden sur les semi-groups répetitifs*, J. Combin. Theory Ser. A **12** (1972), 357–367.

[4] R. Rado, *Axiomatic treatment of rank in infinite sets*, Canad. J. Math. **1** (1949), 337–343.