Common Transversals

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Abstract

Given t families, each family consisting of s finite sets, we show that if the families "separate points" in a natural way, and if the union of all the sets in all the families contains more than $(s+1)^t - s^{t-1} - 1$ elements, then a common transversal of the t families exists. In case each family is a covering family, the bound is $s^t - s^{t-1}$. Both of these bounds are best possible. This work extends recent work of Longyear [2].

1 Introduction and statement of results

Throughout this paper, the symbol $\mathcal{F}$ will always denote a family of t families of sets, each of the t families consisting of s finite, but not necessarily distinct or nonempty, sets. The symbol W will always denote the union of all of the sets contained in all of the t families. Thus $\mathcal{F} = (F_1, F_2, \ldots, F_t)$, where for each j, $1 \leq j \leq t$, $F_j = (F_j(1), F_j(2), \ldots, F_j(s))$ is a family of s (finite, but not necessarily distinct or nonempty) sets, and $W = \bigcup \{F_j(i) : 1 \leq j \leq t, 1 \leq i \leq s\}$ (or more briefly, $W = \bigcup\bigcup \mathcal{F}$). We always assume that $\mathcal{F}$ *separates points* of W in the following sense. Letting $F_j(0) = W - \bigcup F_j$, $1 \leq j \leq t$, we require

$$\left| \bigcap F_j(a \right.$$

(a) If $|W| > s^t - s^{t-1}$ *then each family* $\mathcal{F}_j$ *has a transversal, and* $s^t - s^{t-1}$ *is best possible.*

(b) If $|W| > s^t - s^{t-1}$ *then a common transversal of* $F_1, F_2, \ldots, F_t$ *exists, and* $s^t - s^{t-1}$ *is best possible.*

Theorem 2. *Let* $\mathcal{F}$ *and* W *be as in the first paragraph of this paper.*

(a) If $|W| > (s+1)^t - (s+1)^{t-1} - 1$ *then each family* $\mathcal{F}_j$ *has a transversal, and* $(s+1)^t - (s+1)^{t-1} - 1$ *is best possible.*

(b) If $|W| > (s+1)^t - s^{t-1} - 1$ *then a common transversal of* $F_1, F_2, \ldots, F_t$ *exists, and* $(s+1)^t - s^{t-1} - 1$ *is best possible.*

2 Proofs

Let us show first of all that the bounds given in Theorems 1 and 2 are best possible.

For Theorem 1(a) and Theorem 1(b) let

$$W = \{(a_1, a_2, \ldots, a_t) : 1 \le a_j \le s, 1 \le j \le t-1, 1 \le a_t \le s-1\}.$$

For all j, i, $1 \le j \le t$, $1 \le i \le s$, let $F_j(i) = \{w \in W : \text{the } j\text{th coordinate of } w \text{ equals } i\}$. Note that $F_t(s) = \emptyset$, so that $\mathcal{F}_t$ has no transversal. It is easy to see that $|W| = s^t - s^{t-1}$, each $\mathcal{F}_j$ covers W, $1 \le j \le t$, and $\mathcal{F} = (\mathcal{F}_1, \mathcal{F}_2, \ldots, \mathcal{F}_t)$ separates points.

For Theorem 2(a), we let

$$W = \{(a_1, a_2, \ldots, a_t) : 0 \le a_j \le s, 1 \le j \le t-1, 0 \le a_t \le s-1\} \setminus \{(0, 0, \ldots, 0)\}.$$

For all j, i, $1 \le j \le t$, $1 \le i \le s$, let

$$F_j(i) = \{w \in W : \text{the } j\text{th coordinate of } w \text{ equals } i\}.$$

Again $F_t(s) = \emptyset$ so $\mathcal{F}_t$ has no transversal, and it is easy to see that $|W| = (s+1)^t - (s+1)^{t-1} - 1$, $W = \bigcup\bigcup \mathcal{F}$, and $\mathcal{F} = (\mathcal{F}_1, \mathcal{F}_2, \ldots, \mathcal{F}_t)$ separates points.

For Theorem 2(b), let W be the set of all t-tuples $(a_1, a_2, \ldots, a_t)$, $0 \le a_j \le s$, $1 \le j \le t$, *excluding* the set $(\{(a_1, a_2, \ldots, a_t) : 1 \le a_j \le s, 1 \le j \le t-1, a_t = s\} \cup \{(0, 0, \ldots, 0)\})$. For all j, i, $1 \le j \le t$, $1 \le i \le s$, let

$$F_j(i) = \{w \in W : \text{the } j\text{th coordinate of } w \text{ equals } i\}.$$

Then any element w of $F_t(s)$ must have its j_0th coordinate equal to 0 for some $j_0 \ne t$, and hence w cannot represent any set in the family $\mathcal{F}_{j_0}$, therefore w cannot belong to any common transversal of $F_1, F_2, \ldots, F_t$. Therefore no common transversal exists. Again it is easy to see that $|W| = (s+1)^t - s^{t-1} - 1$, $W = \bigcup\bigcup \mathcal{F}$, and $\mathcal{F} = (\mathcal{F}_1, \mathcal{F}_2, \ldots, \mathcal{F}_t)$ separates points.

Throughout the remaining proofs, the following notation will be used. It is therefore fixed once and for all. For $t \ge 2$, let X be the set of all $(t-1)$-tuples $(a_1, a_2, \ldots, a_{t-1})$, where each a_j, $1 \le j \le t-1$ satisfies $1 \le a_j \le s$. Note that $|X| = s^{t-1}$. For each $x = (a_1, a_2, \ldots, a_{t-1}) \in X$, we denote by $f(x)$ the set $\bigcap\{F_j(a_j) : 1 \le j \le t-1\}$. Then since $\mathcal{F}$ distinguishes points and each $\mathcal{F}_j$ covers W we have $|f(x) \cap F_t(i)| \le 1$ for all $x \in X$ and all i, $1 \le i \le s$, and $W = \bigcup\{f(x) : x \in X\}$.

Proof of Theorem 1(a). The case $t = 1$ follows from the various definitions, so we assume $t \geq 2$, and without loss of generality we restrict our attention to F_t. We shall make use of the classical result of P. Hall [1] according to which F_t has a transversal if and only if $\bigcup F_t(i) : i \quad I \quad \geq I$ for all I $1;2;:::;s$. Suppose that F_t does not have a transversal, and that $F_t(i_1) \quad F_t(i_2) \quad \cdots \quad F_t(i_{k-1})$ $(F_t(i_k) = \emptyset$ if $k = 1$), hence $W = \bigcup F_t(i) : 1 \leq i \leq s; i = i_k$. Then

$$
\begin{aligned}
W &= \left| \left(\bigcup f(x) : x \quad X \right) \quad \left(\bigcup F_t(i) : 1 \leq i \leq s; i = i_k \right) \right| \\
&= \left| \bigcup f(x) \quad F_t(i) : x \quad X; 1 \leq i \leq s; i = i_k \right| \\
&\leq \quad (x;i) : x \quad X; 1 \leq i \leq s; i = i_k \\
&= s^{t-1}(s-1) = s^t - s^{t-1};
\end{aligned}
$$

contrary to the hypothesis of the Theorem. Hence F_t and similarly each F_j, has a transversal. □

Proof of Theorem 1(b). Since the family $F_t = (F_t(1); F_t(2); :::; F_t(s))$ has a transversal (by Theorem 1(a)) and covers W, we can replace F_t by a *partition* $G = (G(1); G(2); :::; G(s))$ such that $G(i) \quad F_t(i)$ for all i, $1 \leq i \leq s$. (The partition G can be constructed as follows. Let $w_1; w_2; :::; w_s$ be a transversal of F_t, where $w_i \quad F_t(i)$, $1 \leq i \leq s$. Let

$$
\begin{aligned}
G(1) &= F_t(1) \quad w_2; :::; w_s \ ; \\
G(2) &= F_t(2) \quad (G(1) \quad w_3; :::; w_s \ ); \\
G(3) &= F_t(3) \quad (G(1) \quad G(2) \quad w_4; :::; w_s \ ); \\
&\vdots \\
G(s) &= F_t(s) \quad (G(1) \quad G(2) \quad \cdots \quad G(s-1)):
\end{aligned}
$$

Then $\mathcal{F}' = (F_1; F_2; :::; F_{t-1}; G)$ distinguishes points, hence $f(x) \quad G(i) \ \leq 1$ for all $x \quad X$ and all i, $1 \leq i \leq s$, and any common transversal of $F_1; F_2; :::; F_{t-1}, G$ is a common transversal of $F_1; F_2; :::; F_t$.

At this point we could in fact replace *every* F_j by a partition (since we know by Theorem 1(a) that every F_j has a transversal); however, it is not necessary, and so we do not.

We now demonstrate the existence of a common trasversal of $F_1; F_2; :::; F_{t-1}; G$.

To this end we define a *diagonal* of X to be a subset D of X such that $D = s$ and for each j, $1 \leq j \leq t-1$, the jth coordinates of the elements of D run through $1;2;:::;s$ in some order. Note that whenever $D = \ x_1; x_2; :::; x_s$ is a diagonal, $w_k \quad f(x_k)$, $1 \leq k \leq s$, and $w_1; w_2; :::; w_s$ are all distinct, then $w_1; w_2; :::; w_s$ is a common transversal of $F_1; F_2; :::; F_{t-1}$.

Now we let $\mathcal{D}$ be some fixed collection of *mutually disjoint* diagonals of X whose union is X, $X = \bigcup \mathcal{D}$. (The existence of $\mathcal{D}$ can be shown by induction on t.)

Since $X = s^{t-1}$ and every diagonal has s elements, we have $\mathcal{D} = s^{t-2}$. For any diagonal D, let $f(D) = \bigcup f(x) : x \quad D$. Then

$$
W = \bigcup f(x) : x \quad X = \bigcup f(D) : D \quad \mathcal{D} :
$$

Now

$$s^t - s^{t-1} < \ \mathrm{W} \le \sum_{D\in\mathscr{D}} f(D) \ \le \ \mathrm{D} \ \max \ f(D) \ : D \quad \mathrm{D} \ ;$$

hence $s^2 - s < \max \ f(D) \ : D \quad \mathrm{D}$.

$G_1; G_2; \ldots; G_t$ is also a common transversal of $F_1; F_2; \ldots; F_t$. Since $\mathcal{F}$ separates points, the cardinal of $W \setminus W_0$ cannot exceed the cardinal of the set of all the t-tuples $(a_1; a_2; \ldots; a_t)$, $0 \le a_j \le s$, having at least one coordinate equal to 0 (excluding $(0; 0; \ldots; 0)$). That is, $|W \setminus W_0| \le (s+1)^t - s^t - 1$. Hence

$$\begin{aligned}(s+1)^t - s^{t-1} - 1 < |W| &= |W_0| + |W \setminus W_0| \\ &\le |W_0| + (s+1)^t - s^t - 1;\end{aligned}$$

and therefore

$$|W_0| > s^t - s^{t-1}:$$

This completes the proof of Theorem 2(b). □

References

[1] Philip Hall, *On representatives of subsets*, J. London Math. Soc. **10** (1935), 26–30.

[2] J.Q. Longyear, *Common transversals in partitioning families*, Discrete Math. **17** (1977), 327–329.