Common Transversals

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Abstract

A 2-coloring of the non-negative integers and a function h are given such that if P is any monochromatic arithmetic progression with first term a and common difference d then $|P| \leq h(a)$ and $|P| \leq h(d)$. In contrast to this the following result is noted. For any k, f there is $n = n(k, f)$ such that whenever n is k-colored there is a monochromatic subset A of n with $|A| > f(d)$, where d

Lemma. f[illegible] $'$ [illegible] $[2^k, 2^{k+1})$ [illegible] $2n-1$ [illegible] $'$ [illegible] in $[2^{k+1}, 2^{k+2})$.

Lemma. [illegible] ≥ 0 [illegible] ≥ 2, [illegible] $' = 2^{p+1}$ [illegible] $\in [2^p, 2^{p+1})$, d $' =$ [illegible] $= 0$. [illegible] $m \in w$, [illegible] $'$ [illegible] $+$ [illegible] $'$

3 The positive result concerning sets with given gap size

Although the results noted here are not new, Fact 3 provides an interesting contrast to the negative result above. The proofs are omitted. Fact 3, the finite version of Fact 2, can be proved by a simple induction on the number of colors.

Let $A = \{a_1, \dots, a_m\}$ be a finite subset of ω, with $a_1 < \dots < a_m$. Define $g(A)$, the "gap size" of A, by $g(A) = \max\{a_{j+1} - a_j : 1 \le j < m\}$ if $|A| > 1$ and $g(A) = 1$ if $|A| = 1$.

Fact 2. [illegible] $k \in \omega$ [illegible] ω [illegible] $\in \omega$ [illegible] $g(A) = d$

Fact 3. [illegible] $k \in \omega, f \in \omega^{\omega}$ [illegible] $|A| > f(g(a))$.

We remark that if $n(k, f)$ is the smallest such n, then $n(1, f) \le 1 + f(1)$ and $n(k, f) \le 1 + kf(n(k-1, f))$. (Letting e denote the identity function, this gives $n(k, e) \le k!(1 + 1/1! + \dots + 1/k!)$, while in fact $n(k, e) = k^2 + 1$; hence the above bound is far from best possible.)

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[illegible] The author completely overlooked Justin's very different construction ([3]– long before the Paris and Harrington result) of a 2-coloring of ω such that any arithmetic progression P with common difference d has $|P|$ bounded by a function of d, namely, for each $n \ge 1$, let $n! = 2^t q$, q odd, and define $c(n) = 0$ if t is even, $c(n) = 1$ if t is odd.

References

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