Common Transversals for Three Partitions

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Abstract

This note contains some questions and a result concerning common transversals for partitions (and in particular for three partitions) of a finite set.

In a famous paper of 1935 [2], Philip Hall gave the first (and still the best!) necessary and sufficient condition for the existence of a system of distinct representatives, or transversal, of a family of sets.

(A set T is a *transversal* of the family $A = (A(1), \dots, A(s))$ if there is a bijection f from $\{1, \dots, s\}$ onto T such that $f(i)$ is an element of $A(i)$, $1 \leq i \leq s$. Hall's Theorem, beautiful in its simplicity, states that if $A = (A(1), \dots, A(s))$ is any family of s sets (not necessarily distinct), then a transversal for the family A exists if and only if the following condition holds: For each k, $1 \leq k \leq s$, the union of any k of the sets $A(i)$ contains at least k elements.)

One of the most singular open questions in transversal theory [4] is the question of whether or not there exists a simple necessary and sufficient condition for the existence of a common transversal for three families.

(If several families of sets are given, say $A_1, \dots, A($ $A(1), = ($ $A(1s))$

Thus let $M(s)$ be the $s\times s\times s$ cube of lattice points defined by $M(s)=\{(a_1,a_2,a_3):0\le a_1,a_2,a_3\le s-1\}$, and let X be a subset of $M(s)$. The *planes of* X are the 3s sets $X\cap\{(a_1,a_2,a_3):a_i=j\}$, $1\le i\le 3$, $0\le j\le s-1$.

What is the maximum size $f(s)$ of a subset X of $M(s)$ such that each plane of X is non-empty and X does not contain any subset T meeting each plane of X in exactly one point?

Taking $X=M(s)\cap\{(x,0,0),(0,y,0),(x,y,z):x\ne 0,y\ne 0\}$ shows that $2(s-1)+s(s-1)^2\le f(s)$. It is also known ([3]) that $f(s)\le s^3-s^2$. Probably one can show that $f(s)=s^3-(2+o(1))s^2$ as $s\to ¥$. Best of all would be to find the exact value of $f(s)$! (the author is inclined to believe that the construction above is “best possible", so that $f(s)=2(s-1)+(s-1)^2s$.)

It is natural to generalize this problem to the t-dimensional “cube" $M(s,t)=\{(a_1,\ldots,a_t):0\le a_i\le s-1, 1\le i\le t\}$. When X is a subset of $M(s,t)$, the *hyperplanes of* X are the sets $X\cap\{(a_1,\ldots,a_t):a_i=j\}$, $1\le i\le t$, $0\le j\le s-1$. What is the maximum size $f(s,t)$ of a subset X of $M(s,t)$ such that each hyperplane of X is non-empty and X does not contain any subset T meeting each hyperplane of X in exactly one point? Is it possible that the computation of $f(s,t)$ for all s,t is an NP-complete problem?

Setting $s=t$, and generalizing the construction above which gives $2(s-1)+(s-1)^2s\le f(s)$ (see [1] for details) leads to the following *conjecture*. For every $e>0$ there exists $n(e)$ such that if $s\ge n(e)$ and X is any subset of $M(s,s)$ with each hyperplane of X containing at least $(1/e+e)s^{-1}$ points, then X contains a subset T meeting each hyperplane of X in exactly one point (where $e=2.718\ldots$).

Other related questions can be found in [1] and [3].

References

[1] T.C. Brown, *Common transversals for partitions of a finite set*, Discrete Math. **51** (1984), 119–124.

[2] Philip Hall, *On representatives of subsets*, J. London Math. Soc. **10** (1935), 26–30.

[3] J.Q. Longyear, *Common transversals in partitioning families*, Discrete Math. **17** (1977), 327–329.

[4] L. Mirsky, *Transversal theory*, Academic Press, New York, 1971.