Some Quantitative Aspects of Szemerédi's Theorem Modulo n

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Abstract

The multiset $P = \{a_1, \dots, a_k\}$ is a k-term arithmetic progression modulo n if $a_1 \not\equiv a_2 \pmod{n}$ and $a_2 - a_1 \equiv a_3 - a_2 \equiv \cdots \equiv a_k - a_{k-1} \pmod{n}$. For k odd and $k \geq 3$, we find explicit constnats $\varepsilon_k < 1 - 1/k$ such that for any $n \neq k$ and for any subset A of $[0, n-1]$, if $|A| > \varepsilon_k n$ then A contains a k-term arithmetic progression modulo n. ($\varepsilon_3 = .5$ and ε_5 is about .77.)

1 Introduction

For each real number $\varepsilon > 0$ and positive integers k and n_0, let $S(\varepsilon, k, n_0)$ denote the following statement.

$S(\varepsilon, k, n_0)$: For every $n \geq n_0$, and for every subset A of $[0, n-1]$, if $|A| > \varepsilon n$ then A contains a k-term arithmetic progression.

Then Szemerédi's theorem [2] asserts that for every $\varepsilon > 0$ and k, there exists a least positive integer $n_0 = n_0(\varepsilon, k)$ such that $S(\varepsilon, k, n_0)$ holds.

One can ask the following quantitative questions. (Answering them, of course, is something else!)

(a) Given $\varepsilon > 0$ and k, what is $n_0[()()]$

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When $m+1 \geq k$, (2) is equivalent to

$$m(m+1) < 2 \cdot \binom{ms-1}{m-1}\binom{ms-2}{m-2} \cdots \binom{ms-k+2}{m-k+2}, \tag{4}$$

and each factor on the right hand side of (4) is greater than s. Therefore when $m+1 \geq k$, (2) holnds provided $m(m+1) \leq 2 \cdot s^{k-2}$, which in turn holds provided $(m+1)^2 \leq 2 \cdot s^{k-2}$, or

$$m \leq \sqrt{2} s^{k/2-1} - 1. \tag{5}$$

Now when $k \leq \sqrt{2} s^{k/2-1}$, we can find an integer m such that $k \leq m+1 \leq \sqrt{2} s^{k/2-1}$ and $m > \sqrt{2} s^{k/2-1} - 2$, which gives (1).

Only a small number of pairs (s,k) have $k > \sqrt{2} s^{k/2-1}$ (namely $(s,k) = (2,3)$, $(2,4)$, $(2,5)$, $(2,6)$, $(3,3)$, $(4,3)$), and these can be checked separately, giving (1) in all cases. $\square$

Theorem 2. *Define the numbers ε_k, for odd $k \geq 3$, as follows. Let $\varepsilon_3 = 1/2$. For $k = 2m+1$, $m \geq 2$, let*

$$\varepsilon_k = 1 - \frac{k+1}{k+2}\left(\sqrt{m^2 + \frac{k+2}{k+1}} - m\right). \tag{6}$$

Then $\varepsilon_k < 1 - 1/k$, and for 9.964 veru78g.9626 Tf 6.365 0 Td [()0 [(1)]TJ/F11 9.9626 Tf 4.982 0 T123n0 m 20.082 [(1)]TJ/F1

For each pair $x, x+y$ ($y \neq 0$) of elements of A, the (distinct) elements $w_1 = x-y$, $w_2 = x+2y$ are excluded from A, since A contains no 3-progression modulo p. (All arithmetic operations here are modulo p.)

Also, given distinct elements w_1, w_2 in $[0, p-1]$, there are unique x, y ($y \neq 0$) in $[0, p-1]$ such that $x - y = w_1$ and $x + 2y = w_2$.

It easily follows that each excluded pair $\{w_1, w_2\}$ is excluded only once, so that the $\binom{\alpha p}{2}$ pairs of elements of A exclude $\binom{\alpha p}{2}$ distinct pairs $\{w_1, w_2\}$ from A. The union of these $\binom{\alpha p}{2}$ distinct pairs of elements has at least αp elements.

Thus $\alpha p = |A| \leq p - \alpha p$, and $\alpha \leq 1/2$, as required. □

Case 2. The case $k > 3$. From now on, for convenience, we abbreviate “k-progression modulo p" to “k-progression".

Let $k = 2m+1$, $m \geq 2$. Let p be prime, $p > k$, $A \subset [0, p-1]$, $|A| = \alpha p$, and assume that A contains no k-progression.

We need to show that $\alpha \leq \varepsilon_k$. (One can check directly that $\varepsilon_k < 1 - 1/k$. ε_5 is about 0.77.)

The argument proceeds essentially as in the case $k = 3$:

Each $(k-1)$-progression contained in A el operations here are

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(When k