On Finitely Generated Idempotent Semigroups

$e \in T_1 \cap T_2$. Then for all $x, y \in T_1 \cup T_2$, as a first step

$$e(xy)e = (eye)(xy)(exe) = e(yexyex)e = e(yex)e = (eye)(exe) = ee = e$$

If $x, y, z \in T_1 \cup T_2$, then using $e(xy)e = e$,

$$xyz = (xex)(yeey)(zez) = x(exye)(eyze)z = xeez = x(exze)z = (xex)(zez) = xz$$

Hence by the maximality of T_1, T_2 we have $T_1 = T_1 \cup T_2 = T_2$. $\square$

Let the equivalence relation on S corresponding to the partition in Lemma 2.2 be denoted by $\sim$, and let the equivalence class containing an element x of S be denoted by T_x. Thus for $x, y \in S$, $T_x = T_y$ $x \sim y$ [both x and y belong to some maximal element of   ].

Lemma 2.3. *For all $x, y \in S$, $T_x = T_y$ [$xyx = x$ and $yxy = y$].*

Proof. One direction is trivial. Suppose now that $xyx = x$ and $yxy = y$. Then the set $\{x, y\}$ satisfies the identity $xyz = xz$, so by Lemma 2.1 can be extended to a maximal element of   . $\square$

Lemma 2.3 shows that $T_x = T_y$ $SxS \cup \{x\} = SyS \cup \{y\}$, so that the sets T_x are the J-classes of S.

Lemma 2.4. *For all $x, y \in S$, $xy \sim yx$. Furthermore, $\sim$ is a congruence on S, that is, for all $x, y, x', y' \in S$, if $x \sim x'$ and $y \sim y'$ then $xy \sim x'y'$.*

Proof. Clearly $(xy)(yx)(xy) = xy$ and $(yx)(xy)(yx) = yx$, hence by Lemma 2.3, $xy \sim yx$. Now assume that $x \sim x'$ and $y \sim y'$. Then

$$xy = (xx'x)(yy'y) \sim (x'xx')(y'yy') = x'y'$$

$\square$

Let $Q(S) = S/\sim$.,3T [(0)]TJETqTd [(s1G [-32c4 0 Td [(Q)]TJ/F8 9.9626 Tf 7.193 0 Td [(()]TJ/F83 9.9626 Tf 3.866 0 Td [(

Theorem 2.1. *For all $n \geq 1$, every idempotent semigroup S on n generators is finite.*

Proof. Let S have generators $g_1, g_2, \ldots, g_n$. The proof has just two ingredients. The first is the fact that if $C(x) = C(z) = \{g_1, g_2, \ldots, g_n\}$, then for any $y \in S$, $C(x) = C(yz) = C(z)$, so that $x, yz, z \in T_x$ and $xyz = x(yz)z = xz$. The second ingredient is a natural definition of the *length* of an element of S. The length of an element of S is defined below, and the proof of the theorem is then identical with the proof below. □

3 The Second Proof

Let S be an idempotent semigroup generated by $g_1, g_2, \ldots, g_n$. Let us call an element $x \in S$ *complete* if for each i, $1 \leq i \leq k$, there are elements a_i and b_i of S such that $x = a_i g_i b_i$. For example, $x = g_1 g_2 \cdots g_n$ is complete. For each $x \in S$, the *length* of x, denoted by $|x|$, is the minimum k such that $x = x_1 x_2 \cdots x_k$, where $x_i \in \{g_1, g_2, \ldots, g_n\}$, $1 \leq i \leq k$. Note that $|x| \geq 1$ for all $x \in S$.

Lemma 3.1. *If $w \in S$ and w is complete, then $w = wxw$ for all $x \in S$.*

Proof. Let $w \in S$ be complete. We show that $w = wxw$ for all $x \in S$ by induction on $|x|$. If $|x| = 1$ then $w = axb$ since w is complete, and

$$\begin{aligned} w &= (ax)b = (axax)b \\ &= a(xaxb) = a(xaxbxaxb) \\ &= (axax)bxaxb = (ax)bxaxb \\ &= (axb)x(axb) = wxw \end{aligned}$$

For the induction step, let $|x| > 1$ and assume that $w = wyw$ for all $y \in S$ with $|y| < |x|$. Let $x = yz$, where $|y| < |x|$ and $|z| < |x|$. Then $w = wyw$ and $w = wzw$, so

$$\begin{aligned} w &= ww = (wzw)wyw) = wzwyw \\ &= w(zwy)w = w(zwyzwy)w \\ &= (wzw)yz(wyw) = wyzw = wxw \end{aligned}$$

□

)wxw

Lemma 3.2. *If $x, y, z \in S_n$, and x, z are complete, then $xyz = xzx$*

z. By Lemma 3.2, $w = xyz = xz$, so $|w| \leq |x| + |z| = 2(t_{n+1} + 1)$, a contradiction. Hence t_n exists and $t_n \leq 2(t_{n-1} + 1)$. □

Department of Mathematics
Simon Fraser University
Burnaby, BC, V6G 1G4
Canada
tbrown@sfu.ca

Department of Mathematics
Southern Illinois University
Edwardsville, IL 62026
USA
elazerson@sbcglobal.net

References

[1] T.C. Brown, *On the finiteness of semigroups in which $x^r = r$*, Proc. Cambridge Philos. Soc. **60** (1964), 1028–1029.

[2] J.A. Green and D. Rees, *On semi-groups in which $x^r = x$*, Proc. Cambridge Philos. Soc. **48** (1952), 35–40.

[3] Gérard Lallement, *Semigroups and combinatorial applications*, Pure and Applied Mathematics. A Wiley-interscience Publication, John Wiley & Sons, New York-Chichester-Brisbane, 1979.

[4] Earl Lazerson, *Idempotent semigroups*, Working Paper No. 38, Communications Research Division, Institute for Defense Analyses, von Neumann Hall, Princeton, NJ, 1961.

[5] M. Lothaire (ed.), *Combinatorics on words*, Cambridge University Press, Cambridge Mathematical Library, Cambridge, 1997, Corrected reprint of the 1983 original, with a new preface by Perrin.

[6] David McLean, *Idempotent semigroups*, Amer. Math. Monthly **61** (1954), 110–113.