Bounds on some van der Waerden numbers

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Abstract

For positive integers s and $k_1,k_2,\ldots,k_s$, the van der Waerden number $w(k_1,k_2,\ldots,k_s;s)$ is the minimum integer n such that for every s-coloring of set $\{1,2,\ldots,n\}$, with colors $1,2,\ldots,s$, there is a k_i-term arithmetic progression of color i for some i. We give an asymptotic lower bound for $w(k,m;2)$ for fixed m. We include a table of values of $w(k,3;2)$ that are very close to this lower bound for $m=3$. We also give a lower bound for $w(k,k,\ldots,k;s)$ that slightly improves previously-known bounds. Upper bounds for $w(k,4;2)$ and $w(4,4,\ldots,4;s)$ are also provided.

1 Introduction

Two fundamental theorems in combinatorics are van der Waerden's theorem [18] and Ramsey's theorem [16]. The theorem of van der Waerden says that for all positive integers s and $k_1,k_2,\ldots,k_s$, there exists a least positive integer $n=w(k_1,k_2,\ldots,k_s;s)$ such that whenever $[1,n]=\{1,2,\ldots,n\}$ is s-colored (i.e., partitioned into s sets), there is a k_i-term arithmetic progression with color i for some i, $1 \le i \le s$.

Similarly, Ramsey's theorem has an associated "threshold" function $R(k_1,k_2,\ldots,k_s;s)$ (which we will not define here). The order of magnitude of $R(k,3;2)$ is known to be $\frac{k^2}{\log k}$ [11], while the best known upper bound on $R(k,m;2)$ is fairly close to the best known lower bound. In contrast, the order of magnitude of $w(k,3;2)$ is not known, and the best known lower and upper bounds on $w(k,k;2)$

of previously known values $w(3,3;2) = 9$, $w(4,4;2) = 35$, and $w(5,5;2) = 178$. A list of other known exact values of $w(k,m;2)$ appears in [15]. Improved lower bounds on several specific values of $w(k,k;s)$ are given in [3] and [10].

In another direction, Graham [7] gives an elegant proof that if one defines $w_1(k,3)$ to be the least n such that every 2-coloring of $[1,n]$ gives either k consecutive integers in the first color or a 3-term arithmetic progression in the second color, then

$$k^{c\log k} < w_1(k,3) < k^{dk^2},$$

for suitable constants $c,d > 0$. This immediately gives $w(k,3;2 < k^{dk^2}$ since we trivially have $w(k,3;2) \leq w_1(k,3)$. In view of Graham's bounds on $w_1(k,3)$, it would be desirable to obtain improved bounds on $w(k,3;2)$. Of particular interest is the question of whether or not there is a non-polynomial lower bound for $w(k,3;2)$.

In this note we give a lower bound of $w(k,3;2) > k^{(2-o(1))}$. Although this may seem weak, we do know that $w(k,3;2) < k^2$ for $5 \leq k \leq 16$ (i.e., for all known values of $w(k,3;2)$ with $k \geq 5$; see Table 2). More generally, we give a lower bound on $w(k,m;2)$ for arbitrary fixed m. We also present a lower bound for the classical van der Waerden numbers $w(k,k,\ldots,k;s)$ that is a slight improvement over previously published bounds. In addition, we present an upper bound for $w(k,4;2)$ and an upper bound for $w(4,4,\ldots,4;s)$.

2 Upper and lower bounds for certain van der Waerden functions

We shall need several definitions, which we collect here.

For positive integers k and n,

$$r_k(n) = \max_{S\subseteq[1,n]} \{|S| : S \text{ contains no } k\text{-term arithmetic progression}\}.$$

For positive integers k and m, denote by $c_k(m)$ the minimum number of colors required to color $[1,m]$ so that there is no monochromatic k-term arithmetic progression.

The function $w_1(k,3)$ has been defined in Section 1. Similarly, we define $w_1(k,4)$ to be the least n such that every 2-coloring of $[1,n]$ has either k consecutive integers in the first color or a 4-term arithmetic progression in the second color.

We begin with an upper bound for $w_1(k,4)$. The proof is essentially the same as the proof given by Graham [7] of an upper bound for $w_1(k,3)$. For completeness, we include the proof here. We will make use of a recent result of Green and Tao [9], who showed that for some constant $c > 0$,

$$r_4(n) < ne^{-c\sqrt{\log\log n}} \tag{1}$$

for all $n \geq 3$.

Proposition 1. *There exists a constant $c > 0$ such that $w_1(k,4) < e^{k^{c\log k}}$ for all $k \geq 2$.*

Proof. Suppose we have a 2-coloring of $[1,n]$ (assume $n \geq 4$) with no 4-term arithmetic progression of the second color and no k consecutive integers of the first color. Let $t_1 < t_2 < \quad < t_m$ be the integers of the second color. Hence, $m < r_4(n)$. Let us define $t_0 = 0$ and $t_{m+1} = n$. Then there must be some i, $1 \quad i \quad m$, such that

$$t_{i+1} \quad t_i > \frac{n}{2r_4(n)}:$$

(Otherwise, using $r_4(n) \quad 3$, we would have $n = \sum_{i=0}^{m}(t_{i+1} \quad t_i) \quad \frac{n(m+1)}{2r_4(n)} \quad \frac{n(r_4(n)+1)}{2r_4(n)} \quad \frac{n}{2} + \frac{n}{6}$.)

Using (1), we now have an i with

$$t_{i+1} \quad t_i > \frac{n}{2r_4(n)} > \frac{1}{2}e^{c^{p}\overline{\log\log n}}:$$

If $n \quad e^{k^{d\log k}}$, $d = c \quad {}^2$, then $\frac{1}{2}e^{c^{p}\overline{\log\log n}} \quad k$ and we have k consecutive integers of the first color, a contradiction. Hence, $n < e^{k^{d\log k}}$ and we are done. □

Clearly $w(k;4;2) \quad w_1(k;4)$. Consequently, we have the following result.

Corollary 2. *There exists a constant $d > 0$ such that $w(k;4;2) < e^{k^{d\log k}}$ for all $k \quad 2$.*

Using Green and Tao's result, it is not difficult to obtain an upper bound for $w(4;4;:::;4;s)$.

Proposition 3. *There exists a constant $d > 0$ such that $w(4;4;:::;4;s) < e^{s^{d\log s}}$ for all $s \quad 2$.*

Proof. Consider a $c_4(m)$-coloring of $[1;m]$ for which there is no monochromatic 4-term arithmetic progression. Some color must be used at least $\frac{m}{c_4(m)}$ times, and hence $\frac{m}{c_4(m)} \quad r_4(m)$ so that $\frac{m}{r_4(m)} \quad c_4(m)$. Let $c > 0$ be such that (1) holds for all $n \quad 3$, and let $m = e^{s^{d\log s}}$, where $d = c \quad {}^2$. Then $c_4(m) \quad \frac{m}{r_4(m)} > e^{c^{p}\overline{\log\log m}} = s$. This means that every s-coloring of $[1;m]$ has a monochromatic 4-term arithmetic progression. Since $m = e^{s^{d\log s}}$, the proof is complete. □

It is interesting that the bounds in Corollary 2 and Proposition 3 have the same form.

The following theorem gives a lower bound on $w(k;k;:::;k;s)$. It is deduced without too much difficulty from the Symmetric Hypergraph Theorem as it appears in [8], combined with an old result of Rankin [17]. To the best of our knowledgg

Proof. We make use of the observation that for positive integers s and m, if $s \geq c_k(m)$, then $w(k,k,\ldots,k;s) > m$, which is clear from the definitions. For large enough m, (2) gives

$$c_k(m) < \frac{2m\log m}{r_k(m)}\left(1+\frac{1}{2}\right) = \frac{3m\log m}{r_k(m)}. \tag{4}$$

Now let $d = \left(\frac{1}{2c}\right)^{z+1}$, where c is from (3), and let $m = s^{d(\log s)^z}$, where s is large enough so that (4) holds. By (3), noting that $\log m = d(\log s)^{z+1} = \left(\frac{\log s}{2c}\right)^{z+1}$, we have

$$\frac{m}{r_k(m)} < e^{c(\log m)^{\frac{1}{z+1}}} = e^{c\,\frac{\log s}{2c}} = \sqrt{s}.$$

Therefore,

$$\frac{3m\log m}{r_k(m)} < 3d\sqrt{s}(\log s)^{z+1} < s$$

for sufficiently large s. Thus, for sufficiently large s,

$$c_k(m) < \frac{3m\log m}{r_k(m)} < s.$$

According to the observation at the beginning of the proof, this implies that $w(k,k,\ldots,k;s) > m = s^{d(\log s)^z}$, as required. □

We now give a lower bound on $w(k,m;2)$. We make use of the Lovász Local Lemma (see [8] for a proof), which will be implicitly stated in the proof.

Theorem 5. *Let $m \geq 3$ be fixed. Then for all sufficiently large k,*

$$w(k,m;2) > k^{(m-1)\left(1-\frac{1}{\log\log k}\right)}.$$

Proof. Given m, choose $k > m$ large enough so that

$$k^{\frac{1}{2m\log\log k}} > \left(m - \frac{1}{2\log\log k}\right)\log k \tag{5}$$

and

$$6 < \frac{\log k}{\log\log k}. \tag{6}$$

Next, let $n = \left\lfloor k^{(m-1)\left(1-\frac{1}{\log\log k}\right)} \right\rfloor$. To prove the theorem, we will show that there exists a (red, blue)-coloring of $[1,n]$ for which there is no red k-term arithmetic progression and no blue m-term arithmetic progression.

For the purpose of using the Lovász Local Lemma, randomly color $[1,n]$ in the following way. For each $i \in [1,n]$, color i red with probability $p = 1 - k^{a-1}$ where

$$a := \frac{1}{2m\log\log k},$$

and color it blue with probability $1-p$.

Let P be any k-term arithmetic progression. Then, since $1+x \leq e^x$, the probability that P is red is

$$p^k = (1-k^{a-1})^k \leq e^{-k^{a-1}\cdot k} = e^{-k^a}.$$

Hence, applying (5), we have

$$p^k < \left(\frac{1}{e}\right)^{\left(m-\frac{1}{2\log\log k}\right)\log k} = \frac{1}{k^{m-\frac{1}{2\log\log k}}}.$$

Also, for any m-term arithmetic progression Q, the probability that Q is blue is

$$(1-p)^m = (k^{a-1})^m = \frac{1}{k^{m-\frac{1}{2\log\log k}}}.$$

Now let $P_1, P_2, \dots, P_t$ be all of the arithmetic progressions in $[1,n]$ with length k or m. So that we may apply the Lovász Local Lemma, we form the "dependency graph" G by setting $V(G) = \{P_1, P_2, \dots, P_t\}$ and $E(G) = \{\{P_i, P_j\} : i \neq j, P_i \cap P_j \neq \emptyset\}$. For each $P_i \in V(G)$, let $d(P_i)$ denote the degree of the vertex P_i in G, i.e., $|\{e \in E(G) : P_i \in e\}|$. We now estimate $d(P_i)$ from above. Let $x \in [1,n]$. The number of k-term arithmetic progressions P in $[1,n]$ that contain x is bounded above by $k \cdot \frac{n}{k-1}$, since there are k positions that x may occupy in P and since the gap size of P cannot exceed $\frac{n}{k-1}$. Similarly, the number of m-term arithmetic progressions Q in $[1,n]$ that contain x is bounded above by $m \cdot \frac{n}{m-1}$.

Let P_i be any k-term arithmetic progression contained in $[1,n]$. The total number of k-term arithmetic progressions P and m-term arithmetic progressions Q in $[1,n]$ that may have nonempty intersection with P_i is bounded above by

$$k\cdot\left(k\cdot\frac{n}{k-1}+m\cdot\frac{n}{m-1}\right) < kn\cdot\left(2+\frac{2}{m-1}\right), \tag{7}$$

since $k > m$. Thus, $d(P_i) < kn(2+\frac{2}{m-1})$ when $|P_i| = k$. Likewise, $d(P_i) < mn(2+\frac{2}{m-1})$ when $|P_i| = m$. Thus, for all vertices P_i of G, we have $d(P_i) < kn(2+\frac{2}{m-1})$.

To finish setting up the hypotheses for the Lov'asz Local Lemma, we let X_i denote the event that the arithmetic progression P_i is red if $|P_i| = k$, or blue if $|P_i| = m$, and we let $d = \max_{1 \leq i \leq t} d(P_i)$. We have seen above that for all i, $1 \leq i \leq t$, the probability that X_i occurs is at most

$$q := \frac{1}{k^{m-\frac{1}{2\log\log k}}},$$

while from (7) we have $d < 2kn(1+\frac{1}{m-1})$.

We are nowww.1

Table 1: Small values of $w(k;3)$ and $w_1(k;3)$

k	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
$w(k;3;2)$	6	9	18	22	32	46	58	77	97	114	135	160	186	218	238
$w_1(k;3)$	9	23	34	73	113	193	?	?	?	?	?	?	?	?	?

$d < 3kn$, so that $d+1 < 3kn+$

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