# Arithmetic Progressions in Lacunary Sets

T. C. Brown and A. R. Freedman

**Citation data:** T.C. Brown and A.R. Freedman, *Arithmetic progressions in lacunary sets*, Rocky Mountain J. Math. **17** (1987), no. 3, 587–596.

**Abstract**

We make some observations concerning the conjecture of Erdős that if the sum of the reciprocals of a set $A$ of positive integers diverges, then $A$ contains arbitrarily long arithmetic progressions. We show, for example, that one can assume without loss of generality that $A$ is lacunary. We also show that several special cases of the conjecture are true.

## 1 Introduction

The now famous theorem of Szemerédi [7] is often stated:

*(a) If the density of a set A of natural numbers is positive, then A contains arbitrarily long arithmetic*

Szemerédi actually proved:

*(b) If* $\overline{u}(A) > 0$*, then* $A$ *is* $\omega$*-good.* Hence we also have

*(c) If* $\overline{\delta}(A) > 0$*, then* $A$ *is* $\omega$*-good.*

In fact, Szemerédi proved the following "finite" result (which we state in a general form to be used later):

*(d) Let* $\varepsilon > 0$ *and* $k \in N = \{1, 2, 3, \ldots\}$*. Then there exists an* $n_0 \in N$ *such that if* $P$ *is any arithmetic progression of length* $|P| \geq n_0$ *and* $A \subseteq P$ *with* $|A| \geq \varepsilon |P|$*, then* $A$ *is* $k$*-good.*

It is not hard to prove (without assuming the truth of any of the statements) that (b), (c), and (d) are equivalent.

Erdős has conjectured that the following stronger statement holds:

*(e) If* $A \subseteq N$ *and* $\sum_A \frac{1}{a} = \infty$*, then* $A$ *is* $\omega$*-good.*

By $\sum_A (1/a)$ we mean of course $\sum_{a \in A} (1/a)$. The proof (or disproof) of (e) is, at present, out of sight. In fact, it has not even been proved that $\sum_A (1/a) = \infty$ implies that $A$ is 3-good (compare Roth [6]). That (e) $\Rightarrow$ (c) can be seen as follows: If $\overline{\delta}(A) = \varepsilon > 0$, then there exists a sequence of natural numbers $0 = n_0 < n_1 < n_2 < \cdots$, such that, for each $i$,

$$\frac{A[1; n_i]}{n_i} > \varepsilon$$

We may assume 3. Suppose the theorem is false. We will construct a set such that å (1= ) = ¥ and is not -good. Choose a nite set $_0$ such that $_0$ is not -good and å (1= ) > 1. Let $_1$ be prime, $_1$ > 2 max $_0$, and choose a nite subset $_1$ of f $_1$ : 1g such that $_1$ is not -good and å $_1$(1= ) > 1. Let $_2$ be prime, $_2$ > 2 max $_1$, and choose a nite subset $_2$ of f $_2$ : 1g such that $_2$ is not -good and å $_2$(1= ) > 1. Continuing in this way, we obtain nite sets $_0$; $_1$; : : : such that for each 0, is not -good, min $_{+1}$ $_{+1}$ > 2 max , each elements of $_{+1}$ is a multiple of $_{+1}$, and å (1= ) > 1.

Let = S . It is clear that å (1= ) = ¥. To show that is not -good, it suf ces to show that every 3-term arithmetic progression contained in must be contained in a single set

To this end, suppose that < < , with ; ; 2 and = . Let 2 . Then 2 also, since otherwise min $_{+1}$ max > max > . Thus ; 2 f : 1g. Hence is divisible by , so > max $_1$, and 2 . This nishes the proof of Theorem 1. □

Corollary 1. ε λλ ω ε ε ε λε ε ε ε ε 2 ε ε ε 2 å (1= ) > ε δ

We state next a lemma which will be useful later.

Lemma 1. ε $_1$; $_2$; : : : ε ε ε ε ε ε ε δ δ min $_{+1}$ 2 max ε = S ( + 1) δ

(The proof of Lemma 1 is contained in the proof of Theorem 1 above).

(2.2). Now we de ne an increasing sequence $_1$ < $_2$ < $_3$ < , of natural numbers to be lacunary if δ = $_{+1}$ ! ¥ as ! ¥ and to be -lacunary if, furthermore, δ δ $_{+1}$ for all . We shall think of such a sequence simultaneously as a sequence and as a subset. Any lacunary sequence has ¯( ) = 0 (see [2]), so that Szemerédi's theorem does not apply.

A subsequence of a lacunary sequence is lacunary, but the corresponding statement, unfortunately, does not hold for -lacunary sequences. It is known that if the real series å is not absolutely convergent, then there exists a lacunary sequence such that å $_2$ diverges (see Freedman and Sember [2]). It follows that if and å (1= ) = ¥, then there exists a lacunary sequence such that å (1= ) = ¥. Thus we have the following.

Theorem 2. ε λλ ω ε ε ε λε ε ε ε λ ε ε ε δ å (1= ) = ¥ ε w δ

Hence we need only investigate lacunary sequences when contemplating the Erdős conjecture. It can also be shown that if å = ¥ and 0 for all , then there exists an -lacunary sequence such that å $_2$ = ¥. (We omit the rather cumbersome proof of this statement.) But notice that this does not imply that statement (h) below is equivalent to statement (e)! This is too bad–because we now prove (h).

Theorem 3. ε λλ ω ε ε ε λ δ å (1= ) = ¥ ε w δ

Let = f $_1$ < $_2$ < $_3$ < g be an -lacunary sequence with in nite reciprocal sum. Assume there is a such that δ < δ$_+$ for each , where δ = $_{+1}$ , 1. We show that $_+$