Cancellation in Semigroups in Which $x^2 = x^3$

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Communicated by G. Lallement

Citation data: T.C. Brown, *Cancellation in semigroups in which $x^2 = x^3$*, Semigroup Forum **41** (1990), 49–53.

1 Introduction

Let $B(k,m,n)$ denote the semigroup generated by k elements and satisfying the identity $x^m = x^n$, where $0 \leq m < n$. That is, $B(k,m,n)$ is the free semigroup on k generators in the variety of semigroups defined by the law $x^m = x^n$ (we are following the notation of Lallement [7]).

Green and Rees [6] showed that for each $n \geq 1$, the semigroups $B(k,1,n)$ are finite for all $k \geq 1$ if and only if the groups $B(k,0,n-1)$ are finite for all $k \geq 1$. Thus in particular all semigroups in which $x = x^2$ are locally finite, and so are all semigroups in which $x = x^3$. The word problem for semigroups in which $x = x^3$ was solved by Gerhard [5].

The existence of an infinite sequence on 3 symbols in which there are no two consecutive indentical blocks shows that $B(3,2,3)$

What could be the largest possible numerical value of such a constant c? This is the subject of the present note.

2 An Upper Bound for the Constant c

Since the value of c depends on k, the number of generators of S, we make the following definition.

Definition. Let $g_1, g_2, \ldots, g_k, \ldots,$ be a sequence so that for each $k \geq 1$, $g_1, g_2, \ldots, g_k$ is a set of generators for the semigroup $B_k = B(k, 2, 3)$, and let $B_w = B_1 \cup B_2 \cup \cdots$. For each $k \geq 1$, let c_k be the largest real number such that for all $x \in B_k$ and all i, $1 \leq i \leq k$, $|g_i x| \geq c_k |x|$. Similarly, let c_w be the largest real number such that for all $x \in B_w$ and all $i \geq 1$, $|g_i x| \geq c_w |x|$.

Note that if C_k is the set of all real numbers c such that $|g_i x| \geq c|x|$ for $1 \leq i \leq k$ and $x \in B_k$, then $c_k = \sup C_k = \max C_k$.

Since $2 = |g_1(g_1)^2| \geq c_1 |(g_1)^2| = 2c_1$, we have $1 = c_1$, and since $B_w \supseteq B_{k+1} \supseteq B_k$, we have $c_k \geq c_{k+1} \geq c_w$, so that

$$1 = c_1 \geq c_2 \geq \cdots \geq c_k \geq c_{k+1} \geq \cdots \geq c_w \geq 0.$$

It is easy to see that $2/3 \geq c_w$. For let $A = g_2 g_3 \cdots g_p$ nd $x = A g_1 A g_1 A$. Then $|x| = 3p - 1$ and $|g_1 x| = |g_1 A g_1 A| = 2p$, so that for all $p \geq 2$,

$$\left(\frac{2}{3} + \frac{2}{9p - 3}\right) |x| = |g_1 x| \geq c_w |x|.$$

B, C, F, G, then $B \approx F$ and $C \approx G$. Then, if the shortest product of generators which equals $ag_1bg_1cg_1$ contains at least three g_1s, it contains only three g_1s, and $|ag_1bg_1cg_1| = |a| + |b| + |c| + 3$. If the shortest such product contains only two g_1s, then it is not hard to see that $a = b = c$. □

Lemma 2. *Define elements x_n, y_n in B_w for all $n \geq 2$ inductively as follows. Let $x_2 = g_2$, $y_2 = g_1g_2$. For $n \geq 2$, let $x_{n+1} = x_ny_ng_{n+1}$, $y_{n+1} = x_ny_n^2g_{n+1}$. Then for $n \geq 2$, $|g_1x_{n+1}y_{n+1}| = |g_1x_ny_n| + |x_ny_n^2| + 2$ and $|x_{n+1}y_{n+1}^2| = |x_ny_n| + 2|x_ny_n^2| + 3$.*

Proof. This follows from Lemma 1, with the g_{n+1} in Lemma 2 playing the role of g_1 in Lemma 1. One needs to know that $x_{n+1} \neq y_{n+1}$. But if $x_{n+1} = y_{n+1}$, then $x_ny_n = x_ny_n^2$, and this implies (by Lemma 1) that $x_{n-1}y_{n-1} = x_{n-1}y_{n-1}^2$. □

Proposition. *Let t denote the golden mean, $t = (1+\sqrt{5})/2 \approx 1.618$. Then $t - 1 \geq c_w$.*

Proof. In our calculation, we will make use of the Fibonacci numbers F_n, where $F_0 = F_1 = 1$ and $F_{n+2} = F_{n+1} + F_n$, and the fact that F_n/F_{n+1} converges to $1/t$.

For $n \geq 2$, let x_n, y_n be defined as in Lemma 2. Then by induction it follows that for all $n \geq 2$, $g_1x_ny_n^2 = g_1x_ny_n$. By Lemma 2 and induction it follows that for all $n \geq 2$, $|g_1x_ny_n| = F_{2n-3} + F_{2n-1}$, $|x_ny_n^2| = F_{2n-2} + F_{2n} - 2$.

Then for all $n \geq 2$,

$$c_w \leq \frac{|g_1x_ny_n^2|}{|x_ny_n^2|} = \frac{|g_1x_ny_n|}{|x_ny_n^2|} = \frac{F_{2n-3} + F_{2n-1}}{F_{2n-2} + F_{2n} - 2} \to 1/t = t - 1,$$

and it follows that $c_w \leq t - 1$. □

3 An Open Question

It would be interesting to know the exact values of c_2 and c_w, and inparticular whether $c_2 > 0$, and whether $c_w > 0$.

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