A Pseudo Upper Bound for the van der Waerden Function *

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Abstract

For each positive integer n, let the set of all 2-colorings of the interval $[1,n] = \{1,2,\dots,n\}$ be given the uniform probability distribution, that is, each of the 2^n colorings is assigned probability 2^{-n}. Let f be any function such that $f(k)/\log k \to \infty$ as $k \to \infty$. For convenience we assume that $f(k)2^k$ is always a positive integer. We show that the probability that a random 2-coloring of $[1, f(k)2^k]$ produces a monochromatic k-term arithmetic progression tends to 1 as $k \to \infty$. We call $f(k)2^k$ a *pseudo upper bound* for the van der Waerden function. We also prove the "density version" of this result.

1 Introduction

Let w denote the van der Waerden function. By definition, for each integer $k \geq 1$, $w(k)$ is the smallest positive integer such that every 2-coloring of the interval $[1; w(k)] = \{1; 2; \dots; w(k)\}$ produces a monochromatic k-term arithmetic progression. (Equivalently, for every partition of $[1; w(k)]$ into at most two parts, at least one part contains a k-term arithmetic progression.)

The existence of $w(k)$, $k \geq 1$, was proved by van der Waerden in 1927 [7]. The best known lower bound for $w(k)$ is $w(k) > (2^k/2ek)(1 + o(1$-term arih 3.896 f 4.982 0 Td [(]TJ)ITd [1002/F8 9.962p26 Tf 4.981 0 Td [(el

To illustrate the method, consider 2-colorings of the interval $[1;tk]$, where $t = k2^k$, and let T_k denote the set of all those 2-colorings of $[1;tk]$ for which none of the t intervals $[1;k]$, $[k+1;2k]$, ..., $[(t-1)k+1;tk]$ is monochromatic. Then under the correspondence $x_0x_1 \cdots x_{s-1} \leftrightarrow \sum_{i=1}^{s-1} x_i k^i$. That is, we identify each integer in $[0; k^s - 1]$ with the s-tuple of the digits in its k-ary expansion.

Under this identification, B_1 may be visualized as the s-dimensional *cube* C, k units on a side. For our purposes, we say that a *line* in the cube C is a set of the form

$$\{x_0 \cdots x_{j-1} y x_{j+1} \cdots x_{s-1} : 0 \le y \le k-1\};$$

where the x_i's are fixed. If the jth coordinate is the "moving" coordinate, then the k points in this line correspond to k integers in B_1 which form an arithmetic progression with common difference k^j.

There are sk^{s-1} lines in the cube C. For each line u in C, let A_i denote the set of 2-colorings of C for which the line u is monochromatic. Then $|A_u| = 2 \cdot 2^{k^s - k}$. Given any two distinct lines u and v, u and v are either disjoint or meet in 1 point. In either case, $|A_u \cap A_v| = 4$

(To see this it is convenient to show first that for any $h > 0$, the inequalities

$$\frac{1}{2}\log(1=e) \quad h \quad \frac{1}{\log k} < \frac{s}{k} < \quad \frac{1}{2}\log(1=e) + h \quad \frac{1}{\log k}$$

hold for all sufficiently large k. For the right-hand inequality, one again needs to assume that $f(k) < k^2$, and handle the case $f(k) > k^2$ by a separate argument, as in the discussion in the Introduction.)

The cube C is defined as before. Let **B** denote a random $ejCj$-element subset of C, where each element of C belongs to **B** with probability e. Let $p_u = \Pr[u \quad$ **B**$] = e^k$, where u is any one of the $sk^{s\ \ 1}$ lines in C, and let $p_{uv} = \Pr[u \quad$ **B** and $v \quad$ **B**$]$, where u and v are distinct lines in C. Then $\Pr[u \quad$ **B** for some $u]$ $\sum_u p_u \quad \sum_{u;v} p_{uv}$.

Through each of the k^s points of C there are s lines, and hence of the $\binom{sk^s\ 1}{2}$ pairs of lines f $u;v$g, exactly $k^s \binom{s}{2}$ pairs meet, and the other pairs are disjoint. Then

$$\sum_u p_u \quad \sum_{u;v} p_{u;v} = \frac{s}{k}k^s e^k \quad k^s \binom{s}{2} \; e^{2k\ \ 1} \qquad \binom{sk^s\ \ 1}{2} \qquad k^s \binom{s}{2} \quad e^{2k}$$

$$= \frac{s}{k}k^s e^k \quad k^s \binom{s}{2} \; e^{2k\ \ 1} \quad \frac{1}{2}\frac{s}{k}k^s e^k \quad \frac{s}{k}k^s e^k \quad e^k \; + k^s \binom{s}{2} \; e^k:$$

The remaining inequalities hold for sufficiently large k.

Since $(s=k)k^s e^k \quad e^k < 1=2$, we get

$$\sum_u p_u \quad \sum_{u;v} p_{u;v} > \frac{3}{4}\frac{s}{k}k^s e^k \quad k^s \binom{s}{2} \; e^{2k\ \ 1}(1 \quad s=k)k^s e^k \quad k^s$$ 1 1(y)]TJ/F131 9.9626 Tf 210.777 where s411 9.9626 Tf 7.087 0 Td [

Perhaps, by using a sufficiently large set of progressions, one could show that $(1+a)^k$ is a pseudo upper bound for the van der Waerden function, for every $a > 0$.

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