Monochromatic structures in colorings of the positive integers and the finite subsets of the positive integers

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Abstract

We discuss van der Waerden's theorem on arithmetic progressions and an extension using Ramsey's theorem, and the canonical versions. We then turn to a result (Theorem 6 below) similar in character to van der Waerden's theorem, applications of Theorem 6, and possible canonical versions of Theorem 6. We mention several open questions involving arithmetic progressions and other types of progressions.

1 van der Waerden's theorem on arithmetic progressions

One of the great results in combinatorics is the following theorem.

Theorem 1. *(van der Waerden's theorem on arithmetic progressions) If N is finitely colored (= finitely partitioned) then some color class (= cell of the partition) contains arbitrarily large arithmetic progressions* $P = \ a, a+d, a+2d, \ldots, a+(n \quad 1)d \ .$

Van der Waerden's original proof is in [30]. The most famous proof (essentially van der Waerden's own proof) is in [20]. See also [31]. The shortest proof is in [18]. The clearest proof is probably in [22]. A topological proof can be found in [15]. For other proofs, see [1, 6, 12, 13, 23–25, 29, 32].

The "canonical" version of van der Waerden's theorem is the following, due to Erdős and Graham [14].

Theorem 2. *Given f Givena2in626 Tf 4.981 00 .962623.362623.3owin626 Tf 4.981 00 .962623.362623.3o;a*

Using the definition of $w(k)$ from the preceding theorem, the only known values of $w(k)$ are $w(1) = 1$, $w(2) = 3$, $w(3) = 9$, $w(4) = 35$, $w(5) = 178$. For values involving more than two colors, see [2,4,11,19,26].

Berlekamp [7] showed in 1961 that $k2^k < w(k+1)$ if k is prime.

Erdős asked in 1961 whether or not $\lim_{k\to ¥} \frac{w(k)}{2^k} = ¥$, and offered US \$25 for an answer. This question is still open, and the prize is still available.

Szabo [28] showed in 1990 that $\frac{2^k}{k^e} < w(k)$, $k > k(e)$.

Gowers showed in 1998 [16,17] that $w(k) < 2^{2^{2^{2^{2^{k+9}}}}}$.

Graham asked in 1998 whether $w(k) < 2^{k^2}$, and offers US \$1000 for an answer.

Using Ramsey's theorem, the following "extended" van der Waerden's theorem can be proved.

Theorem 4. *(Extended van der Waerden's theorem) If $P_f(\mathbb{N})$ (the collection of all finite subsets of $\mathbb{N}$) is finitely colored, then for every $n \quad 1$ there exist an infinite set $Y \quad \mathbb{N}$ (Y depends on n) and an arithmetic progression $a, a+d, a+2d, \ldots, a+(n \quad 1)d$ such that the set $[Y]^a \quad [Y]^{a+d} \quad [Y]^{a+2d} \qquad [Y]^{a+(n-1)d}$ is monochromatic. (Here $[Y]^k$ denotes the set of all k-element subsets of Y.)*

Proof. Let $g : P_f(\mathbb{N}) \quad [1, r]$ be an r-coloring of $P_f(\mathbb{N})$. Let n be given. By van der Waerden's theorem, choose m large enough that every r-coloring of $[1, m]$ produces a monochromatic n-term arithmetic pro-

3. Vertex y at level $i+1$ covers vertex x at level i iff $x \quad y$.

4. If y, z cover x then $y \quad z = x$.

5. Each element of Y appears in at most one tree of the forest F.

6. Each vertex not at level $n \quad 1$ has infinitely many immediate successors.

7. F is the union of infinitely many non-empty trees.

Let us call such a structure an "arithmetic w-forest of height n."

Diana Piguetova, a student of Jarik Nešetřil, has proved the following result.

Theorem 5. *If $g : P_f(\mathbb{N}) \quad w$ is an arbitrary coloring, then for every $n \quad 1$*

3. It has an extremely simple proof, by induction on the number of colors.

4. The d in the conclusion is fixed.

5. No canonical version is known.

An application of Theorem 6 is the following result, proved in [8, 9].

Theorem 7. *Let S,T be semigroups and let $j : S \to T$ be a homomorphism. Assume that T is locally finite, and that for every idempotent e in T, $j^{-1}(e)$ is locally finite. Then S is locally finite.*

(For groups, this is an old theorem of O. Schmidt: A locally finite extension of a locally finite group is locally finite.)

Sketch of proof. Some simple considerations reduce the proof to the following case. Let $j : S \to G$, where G is a finite group, and assume that $j^{-1}(e)$ is locally finite. Assume that S is generated by $W = \{w_1, w_2, \ldots, w_t\}$. It is necessary to show that S is finite. It suffices for this (by a simple compactness argument) to show that every *sequence* $s = x_1x_2x_3\ldots$ of elements of W contains a "contractible" factor $x_{j+1}x_{j+2}\cdots x_{j+k}$, that is, a factor $x_{j+1}x_{j+2}\cdots x_{j+k}$ which equals the product of fewer than k elements of W.

Define the finite coloring f of $\mathbb{N}$ by $f(m) = j(x_1x_2\cdots x_m)$ for all $m \in \mathbb{N}$. Then, by Theorem 6, we have a *fixed* d and, for every n

1952 Green & Rees – [Every group with $x^n = 1$ is locally finite] [Every semigroup with $x^{n+1} = x$ is locally finite.]

1957 M. Hall Jr. – Yes, if $x^6 = 1$.

1964 Golod & Shafarevich – No, if $x^{n(x)} = 1$. (6 pages. See the book *Noncommutative Rings*, by I. N. Herstein, Mathematical Association of America, Washington, DC, 1994.)

1965 Novikov & Adian – No, if $x^n = 1$, for odd n 4381. (300+ pages.)

1975 Adian – No, if $x^n = 1$, for odd n 665.

1992 Lysionok – No, if $x^n = 1$, for all n 213.

(See the book *Around Burnside*, by A. I. Kostrikin, Springer-Verlag, Berlin, 1990.)

3 On the canonical version of Theorem 6

In this section we describe a 2-coloring f of w which shows that the constant colorings and the 1-1 colorings are not sufficient for a canonical version of Theorem 6. That is, there does not exist a fixed d and arbitrarily large sets $A = \quad a_1 < a_2 < a_3 < \quad < a_n$ with max $a_{j+1} \quad a_j \; j = 1, 2, \ldots, n \quad 1 \; = d$, such that f_A is either constant or 1-1. In fact, even the “almost constant” colorings (c colors are allowed, where c is a constant) and the “almost 1-1” colorings (at most c-to-1, where c is a constant) are not enough. (See Theorem 8 below.) We omit the proofs.

Let S denote the set of all sums of distinct even powers of 2, including 0 as the empty sum. Thus $S = \quad 0, 1, 4, 5, 16, 17, 20, 21, 64, \ldots$.

Let T denote the set of all sums of distinct odd powers of 2, including 0 as the empty sum. Thus $T = \quad 0, 2, 8, 10, 32, 34, 40, 42, \ldots$.

Order the elements of S and T so that $S = \quad s_0 < s_1 < \quad$ and $T = \quad t_0 < t_1 < \quad$.

Then, for each j, $f^{-1}(j)$ is defined by $f^{-1}(j) = S + t_j = \quad s + t_j \; s \quad S$. Then

$$
\begin{array}{lllllllllll}
0 & = & f(0) & = & f(1) & = & f(4) & = & f(5) & = & \\
1 & = & f(2) & = & f(3) & = & f(6) & = & f(7) & = & \\
2 & = & f(8) & = & f(9) & = & f(12) & = & f(13) & = & \\
3 & = & f(10) & = & f(11) & = & f(14) & = & f(15) & = & \\
 & \ldots & & & & & & & & &)
\end{array}
$$

(Note that in [13], 4 colors are used, 4 times each. In general, in $[0, 4^k$

The set S above (S = set of sums of distinct even powers of 2) is the “Moser-de Bruij[redacted]
(Neil Sloane’s sequence #A000695. See http://www.oeis.org.) Setting $f = f(0)f(1)f($[redacted]
we have $f = [(0011001122332233)^2(4455445566776677)^2]^2$.

Definition 1. *For* $A = \; a_1 < a_2 < a_3 < \quad < a_n$ *with* $\max \; a_{j+1} \quad a_j \; j = 1, 2, \ldots, n \quad 1 \; = d$, *we* [redacted]
that the gap size of A *is* d, *and write* $gs(A) = d$. *If* $A = 1$, *we set* $gs(A) = 1$.

Theorem 8. *With* f *defined as above, and any* $A \quad w$, $\sqrt{A\,/8gs(A)}$ $f($

Question 4. Is it not hard to show that if $1, 2, \ldots, n^2$ is 2-colored, then there exists a monochromatic set $a_1, a_2, \ldots, a_n$ such that the set of consecutive differences $a_{j+1} \quad a_j\ 1 \quad j \quad n \quad 1$ has at most $\overline{n}$ elements? Can one find a "small" $g(n)$ such that if $1, 2, \ldots, g(n)$ is 2-colored, then there is a monochromatic set $a_1, a_2, \ldots, a_n$ such that the set of consecutive differences $a_{j+1} \quad a_j\ 1 \quad j \quad n \quad 1$ has at most $\log n$ elements? For each k can one find (for arbitrarily large n) $a_1 < a_2 < \quad < a_n$ with $a_{j+1} \quad a_j \quad \log n, 1 \quad j \quad n \quad 1$, such that $a_1, a_2, \ldots, a_n$ contains no k-term arithmetic progression?

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