Volume of a sphere with a hole drilled through its centre.
Andrew DeBenedictis.

In the diagram below a hole is drilled through the centre of the sphere. We know the length h ($2h$ is the height of the removed cylinder) and *nothing else!*. The claim is that the volume of the remaining solid is $\frac{4}{3}\pi h^3$. i.e. the volume of a sphere of radius h!

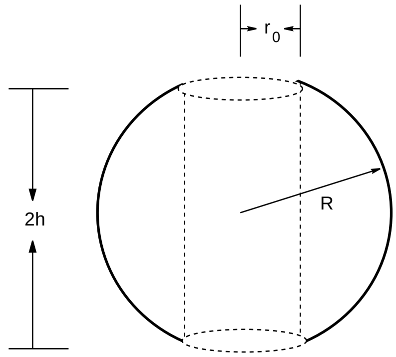

Figure 1: We know the length h and nothing else. What's the volume of the remaining solid?

The volume element in cylindrical coordinates:

$$dV = r\, d\theta\, dr\, dz$$
$$0 \le \theta < 2\pi, \quad r_0 \le r \le \sqrt{R^2 - z^2}, \quad -h \le z \le +h,$$

where R is the radius of the original sphere (a quantity we do not know). The quantity r_0 is the radius of the bored out cylinder:

$$r_0 = \sqrt{R^2 - h^2}. \tag{1}$$

Integrating over the upper-half of the solid and multiplying by two:

$$\begin{aligned} V &= 2\int_{z=0}^{h}\int_{r=r_0}^{\sqrt{R^2-z^2}}\int_{0}^{2\pi} dV \\ &= 2\int_{z=0}^{h}\int_{r=r_0}^{\sqrt{R^2-z^2}}\int_{0}^{2\pi} r\, d\theta\, dr\, dz \\ &= 2\pi h\left(R^2 - \frac{h^2}{3} - r_0^2\right). \end{aligned}$$

By putting in r_0 from (1) we get

$$V = \frac{4}{3}\pi h^3.$$