On the calculation of risk measures for variable annuities with guaranteed benefits

by

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Abstract

With the development of the life insurance industry, different types of life insurance products, in addition to the traditional ones, are being developed. A common and well-known life insurance product is the variable annuity with different types of guaranteed benefit riders, which provides policyholders a high rate of investment return with downside risk protections. Two typical distortion risk measures, VaR (value at risk) and CTE (conditional tail expectation), are widely used to manage insurers' future liabilities to avoid the potential of insolvency. In this project, we consider variable annuities with certain types of guaranteed benefits and various asset price processes, and focus on the calculation of the two risk measures of insurers' net and gross liabilities at the maturity date. Specifically, we consider two types of guaranteed benefit riders, the guaranteed minimum death benefit (GMDB) and the guaranteed minimum maturity benefit (GMMB), and assume that the logarithm of underlying asset returns follows a Cauchy or a skew-normal distribution. Analytical expressions of VaR and CTE for insurers' future liabilities are obtained, and numerical calculation algorithms are proposed. Comparisons of the calculated risk measure results with that under the normal distribution are also presented.

Keywords: Analytical expressions; Risk measures; Variable annuity; Cauchy distribution; Skew-normal distribution.

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Chapter 1

Introduction

1.1 Overview

In recent years, many types of equity-linked life insurance and annuity products have been developed. One common type of equity-linked life insurance product is the variable annuity contract with or without guaranteed benefits. In the U.K. and most of the European countries, variable annuity contracts (commonly used terminology in the U.S.) are called equity-linked policies and, in Canada, they are known as segregated fund policies. In general, variable annuities have a benefit linked to the performance of an investment fund. There are many types of guaranteed benefits associated with variable annuity contracts; we call them rider (or riders if multiple types of guaranteed benefits are applied).

For a variable annuity with guaranteed benefit riders, the policyholder pays the insurer a single premium at the beginning of the policy term or periodic premiums during the policy term. The insurer invests the initial premium and follow-up premiums in an asset during the policy term until they withdraw their money as income. Variable annuities are not absolutely guaranteed with respect to their investment growth; while they allow for huge gains, they also face potential losses. Variable annuities with guaranteed benefits provide policyholders protections against inflation risk and market volatility; for example, a variable annuity with a guaranteed lifetime withdrawal benefit provides the policyholder a guaranteed income for life even if the market remains unstable or drops precipitously. Hence, variable annuities with guaranteed benefits become one of the ideal choices for investors to receive higher expected investment returns with downside financial market protection.

From the insurers' point of view, it is essential to manage and monitor the fund perdistributional models for modeling asset returns in variable annuities with guaranteed benefits. The return models that we consider in this project are the Cauchy distribution and the skew-normal distribution, which could capture the skewness and the heavy tails presented in the data. We follow similar techniques as in Feng and Volkmer (2012) to derive analytical expressions of VaR and CTE for gross liabilities of variable annuities with either GMMB or GMDB rider. We present calculation algorithms based on Monte Carlo simulation for calculating both VaR and CTE risk measures for net liabilities of variable annuities with either rider. The S&P 500 stock index historical data are fitted to the Cauchy and skew-normal models, and numerical values of risk measures under these asset price models are presented. The results for the normal model are also included for comparison purposes.

1.3 Outline

The remainder of this project report is organized as follows. Chapter 2 provides a literature review on the modeling of stock returns, related studies on the pricing and valuation of variable annuities with guaranteed benefits, and risk measure calculation methods. Chapter 3 presents details of three asset models and introduces the concept of future liabilities for a variable annuity with GMMB or GMDB rider. Analytical expressions for gross liabilities are derived, and calculation algorithms for net liabilities are presented under the three asset models. Chapter 4 shows the statistical analysis on the S&P 500 returns data, and the numerical risk measure results under the three fitted asset models. The conclusion of this project and possible further research on related topics are provided in Chapter 5.

Chapter 2

Literature review

In this chapter, we provide a literature review on three topics. We first review the modeling techniques for asset prices, mainly focusing on the application of the lognormal model or stock returns. Thereafter, many studies in financial engineering and actuarial science fields considered other desirable distributions as alternatives to the normal distribution. For example, Eling (2014) fits the stock returns data to some skewed distribution models such as skew-normal and skew-student t distributions; the analysis shows that such skewed distribution models are promising for modeling returns. Choi and Yoon (2020) present model comparison study on several stock returns data by using twelve different distributions, including fat-tail distributions such as the Cauchy distribution, and skewed distributions such as the skew-normal and skew-student t distributions. More recently, Mahdizadeh and Zamanzade (2019) fit the Cauchy distribution to the stock returns data and proposes six new goodness-of-fit tests to show that fat-tail distributions like the Cauchy fit data better than the normal distribution.

2.2 Variable annuity with guaranteed benefits

A variable annuity, also called an equity-linked insurance contract, is a life insurance product that has been common worldwide since 1960s. Hardy (2003) provides a comprehensive guide and detailed information on life insurance products with investment guarantees, including their modeling and risk management. Major benefit riders introduced in this book are guaranteed minimum maturity benefit (GMMB), guaranteed minimum death benefit (GMDB), guaranteed minimum accumulation benefit (GMAB), guaranteed minimum surrender benefit (GMSB), and guaranteed minimum income/withdrawal benefit (GMIB/GMWB). These guaranteed benefit riders are designed to provide policyholders with downside risk protection when markets are in turmoil. There are many studies on different aspects of variable equation with jumps and obtain numerical results by using the back-propagation neural network. Bacinello et al. (2011) present a unifying approach for the valuation of variable annuities with guaranteed bene t riders. The contract values are computed and compared under di erent valuation approaches using the ordinary and least squares Monte Carlo simulation methods. Huang et al. (2022) develop a computationally e cient approach to value expressions for gross liabilities of the variable annuity with either a GMMB rider or a GMDB rider and use Monte Carlo simulation to calculate the two risk measures based on their corresponding net liabilities.

Chapter 3

Future liabilities and Risk measures

In this chapter, we first introduce two types of insurers' future liabilities for variable annuities with either a GMMB or a GMDB rider. The two types of future liabilities are the gross liability and net liability. We then consider two risk measures, VaR and CTE, and use them to evaluate risks with respect to their future liabilities with guaranteed benefit riders under three equity models (normal, Cauchy, and skew-normal).

We first introduce the notation we use in this study.

- G — the guaranteed level of variable annuity. It represents the lowest value that the policyholder will receive at the end of the policy term.
- T

- $_eL_n^0$ — the present value of insurers' net liability for GMMB rider at time 0.
- $_dL_g^0$ — the present value of insurers' gross liability for GMDB rider at time 0.
- $_dL_n^0$ — the present value of insurers' net liability for GMDB rider at time 0.
- $_tp_x$ — the survival probability for a life age x who will survive t years.
- $_{x+t}$ — the force of mortality for a life age x + t.
- $_x$ — the future li[illegible]

The rest of th[illegible]ganized as follo[illegible]asures for gross and net liabilities are i[illegible]details in Section 3.1. Sec[illegible]duces three models for the underlyi[illegible] then presents analytical expres[illegible]measures for gross liabilities un[illegible]diﬀerent models. Section 3.3 provid[illegible]alytical expressions and numer[illegible]ms for computing and approximating [illegible] risk measures.

3.1 R[illegible]sures and future liabilities

As in Fe[illegible]dely applied risk measures[illegible]ntile of a random variable [illegible]cific significance level (0 1). Given the significance level , VaR repr[illegible]ss amount that will not be exceeded with probability . It is helpful for invest[illegible]ure their potential losses and manage their risk capital during the investmen[illegible]E is the conditional expectation of a random variable given that the random va[illegible]ter than its VaR at a given significance level. CTE helps investors to estimate t[illegible]value of losses given that the losses[illegible]given VaR. CTE is a coherent risk[illegible]e VaR is not because it is n[illegible]e.

3.1.1 GMMB [illegible]

A variable annuity [illegible]

Before we formulate the net liabilities, we introduce an additional quantity called the management expense at time t and denoted by M_t. The management expense represents the management fee that is charged continuously by the insurer during the policy term.

Similar to the net liability defined for the GMMB rider case, the net liability for GMDB rider is given by

$$ {}_dL_n^0 = e^{-r\tau_x}(e^{\tau_x}G - F_{\tau_x})_+ I_{\{\tau_x \le T\}} - \int_0^{\tau_x \wedge T} e^{-rs} M_s ds: \tag{3.5} $$

3.1.3 Two risk measures for guaranteed riders

We discuss the two risk measures, VaR and CTE, for both the GMMB and GMDB riders in this subsection. The quantile risk measure with a given significance level α, denoted by V_α, is define as

$$ V_\alpha \equiv \inf\{x : P[L^0 \le x] \ge \alpha\}; \tag{3.6} $$

where L^0 is a general form of insurer's loss, representing the net present value of insurer's future liability at time 0 in this project. Typical values for α are 95% or 99% (Hardy, 2006). The value of V_α estimates the amount that with probability α, the present value of insurer's future liability will not be exceeded.

The conditional tail expectation risk measure with a given significance level α, denoted by CTE_α, is defined as

$$ CTE_\alpha \equiv E[L^0 | L^0 > V_\alpha]: \tag{3.7} $$

Typical values of α for CTE_α are 90%, 95%, or 99% (Hardy, 2006). The value of CTE_α estimates the amount that represents the average amount of insurer's future liabilities when they exceed V_α.

For insurance companies, it is essential to analyze both gross and net liabilities. Gross liabilities give an insurer a good sense to manage liability risks because gross liabilities do not include any future negative cash flow (management fees), while the net liability includes both the future positive cash flow (benefit payout) and the future negative cash flow (management fees). The latter helps the insurer manage both liability risks and asset risks. In this project, we calculate the two risk measures for gross liabilities by using the analytical formulas we derive, and we estimate the two risk measures for net liabilities based on Monte Carlo simulation.

3.2 Analytical results for gross liabilities

We have introduced the definitions of gross liabilities, net liabilities, and two risk measures for variable annuities with either a GMMB rider or a GMDB rider in Section 3.1. In this section, we first present three models for the asset price process. We then review the analytical results for gross liability risk measures based on the normal model used in Feng and Volkmer (2012) and provide analytical expressions for the gross liability risk measures based on the Cauchy and skew-normal models considered in this project. In the last section of this chapter, we provide algorithms for calculating risk measures of net liabilities by using the Cauchy model as an example.

3.2.1 Underlying equity models

As in Feng and Volkmer (2012), we assume that the account value (market value) of a variable annuity at time t, F_t, is described by

$$F_t = F_0 \frac{S_t}{S_0} e^{\ mt}; \qquad 0 \quad t \quad T; \tag{3.8}$$

where S_t is the market value of the underlying asset at time t, F_0 is the initial payment and its cumulative distribution function (cdf) is given by

$$\Phi(x;\mu,\sigma) = \frac{1}{2}\left[1 + \operatorname{erf}\left(\frac{x-\mu}{\sigma\sqrt{2}}\right)\right]; \quad -\infty < x < \infty; \tag{3.11}$$

where erf(x) is the error function given by

$$\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}}\int_0^x e^{-t^2}\,dt.$$

Assume that the returns in the underlying equity fund from time $t-1$ to time t, for $t \in \mathbb{N}^+$, are identical and independent distributed (i.i.d.). We now first write $\ln(S_t/S_0)$ as

$$\ln\left(\frac{S_t}{S_0}\right) = \underbrace{\ln\left(\frac{S_t}{S_{t-1}}\right) + \ln\left(\frac{S_{t-1}}{S_{t-2}}\right) + \cdots + \ln\left(\frac{S_1}{S_0}\right)}_{\text{independent and identically distributed}}. \tag{3.12}$$

Notice that the terms on the right-hand-side of (3.12) are i.i.d. and follow Norm$(\mu;\sigma)$. Applying the properties of the normal distribution, we then have that

$$\ln\left(\frac{S_t}{S_0}\right) \sim \text{Norm}\left(\mu t; \sigma\sqrt{t}\right); \quad t > 0;$$

or

$$\frac{S_t}{S_0} \sim \text{Lognorm}\left(\mu t; \sigma\sqrt{t}\right); \quad t > 0.$$

ˆt

Note that the Cauchy distribution is also symmetric similar to the normal one, but it does not have a mean, a variance or higher moments. The latter characteristic implies that the Cauchy distribution has a fat tail.

Assume that $\ln(S_1 = S_0); \ln(S_2 = S_1); : : : ;$ are i.i.d. and follow Cauchy(;). By applying the properties of the Cauchy distribution, we can similarly get that

$$\ln \frac{S_t}{S_0} \quad \text{Cauchy}(t; \ t \ ); \qquad t > 0;$$

or

$$\frac{S_t}{S_0} \quad \text{Log-Cauchy}(t; \ t \ ); \qquad t > 0:$$

Similar to (3.9), in this case we can write the underlying asset price process $f S_t g_t \ 0$ as

$$S_t = S_0 e^{t + C_t}; \qquad t > 0;$$

where $f C_t g_t \ 0$ is a Cauchy pr091 and pCf]TJ/F49 10.9091 Tf 8.9 0 Td [(;)]TJ/ -40.637 -1..435 Td [(that)-3

Notice that the skew-normal distribution is an asymmetric distribution. The skew-normal distribution becomes normal distribution when $\omega = 0$, and it becomes half normal distribution when $\omega = -\infty$ or ∞.

Assume that $\ln(S_1/S_0), \ln(S_2/S_1), \ldots,$ are i.i.d. and follow skew-norm(μ, σ, ω) distribution. By applying the properties of the skew-normal distribution, we can similarly get

$$\ln\left(\frac{S_t}{S_0}\right) \sim \text{skew-norm}(\mu t, \sigma\sqrt{t}, \omega), \quad t \geq 0;$$

or

$$\frac{S_t}{S_0} \sim \text{Log-skew-norm}(\mu t, \sigma\sqrt{t}, \omega), \quad t \geq 0.$$

Similar to (3.9), the underlying asset price process $\{S_t\}_{t \geq 0}$ can be written as

$$S_t = S_0 e^{\mu t + \sigma M_t}, \quad t \geq 0;$$

where $\{M_t\}_{t \geq 0}$ is a skew-normal process with location parameter 0, scale parameter $\sqrt{t}$, and shape parameter ω; that is, for a fixed t, $M_t \sim$ skew-norm$(0, \sqrt{t}, \omega)$.

3.2.2 Risk measures for variable annuities with a GMMB rider

Before we proceed to the analytical results, we first determine the probability that positive liabilities occur. The insurer only considers the situation where there is a chance for positive future liabilities because negative future liability represents a profit. Considering the gross liability for a variable annuity with a GMMB rider, the probability that no guarantee payment will be made at maturity is given by

$$\alpha_e = 1 - P[G \leq F_T, \tau_x > T].$$

When calculating VaR and CTE risk measures, the significance level α should be chosen to be larger than α_e

Similarly, in the Cauchy model case, the value of $_e$ is given by

$$_e = 1$$

Proposition 3.2. For the three equity return models described in Section 3.2.1, we have the following results for the conditional tail expectation CTE , given that > $_e$, and for GMMB gross liabilities. Note that $_e$ for corresponding normal, Cauchy, and skew-normal models are respectively given in (3.17), (3.18) and (3.19).

(1) Under the normal model, we have

$$\text{CTE} \;=\; e^{\;rT} G \quad {}_T p_x \frac{F_0}{1} \exp f (\quad r \quad m)T + {}^2T{=}2g (\; Z \;\; ; \;\; {}^{p}\overline{T} ; 1);$$

where z is the 100 % percentile of the standard normal distribution with = (1 -)/$_T p_x$, and is the cdf of the standard normal random variable.

(2) Under the Cauchy model, we have

$$\text{CTE} \;=\; e^{\;rT} G \quad {}_T p_x \frac{F_0}{1} \int_{1}^{\ln(\;a)} e^{y} \;\; f_C(y; (\quad r \quad m)T; \; T\,)dy; \qquad (3.21)$$

where a = (e ^{rT} G V)=F$_0$, and f$_C$ is the pdf of the Cauchy distribution with location parameter (r m)T and scale parameter T .

(3) Under the skew-normal model, we have

$$\text{CTE} \;=\; e^{\;rT} G \quad {}_T p_x \frac{F_0}{1} \int_{1}^{\ln(\;a)} e^{y} \;\; g_S(y; (\quad r \quad m)T; \;\; {}^{p}\overline{T}; !\;)dy;$$

where a = (e ^{rT} G V)=F$_0$, and g$_S$ is the pdf of the skew-normal distribution with location parameter (r m)T, scale parameter $^{p}\overline{\;\;}$

e

in which $_tp_x \;\; _{x+t}$ is the density function of the future lifetime of (x). Using (3.8), we can further write $\;_d$ as

(2) Under the Cauchy model, we have

$$1 \quad = \int_0^T {}_tp_x \quad {}_{x+t}F_C \left(\ln \frac{e^{(\quad r)t}G \quad V}{F_0}; (\quad r \quad m)t; \ t \right) dt; \qquad (3.25)$$

where F_C is the cdf of the Cauchy distribution with corresponding parameters given in the brackets.

(3) Under the skew-normal model, we have

$$1 \quad = \int_0^T {}_tp_x \quad {}_{x+t}G_S \left(\ln \frac{e^{(\quad r)t}G \quad V}{F_0}; (\quad r \quad m)t; \ \sqrt{t}; \ ! \right) dt;$$

where G_S is the cdf of the skew-normal distribution with corresponding parameters given in the brackets.

Proof. See Appendix A.3. □

(3) Under the skew-normal model, we have

$$\mathrm{CTE} = \frac{G}{1-\alpha}\int_0^T e^{(\mu-r)t}\,{}_tp_x\,\mu_{x+t}\,G_S\left(\ln\frac{e^{(\mu-r)t}G-V}{F_0};(\mu-r-m)t;\sigma\sqrt{t};\omega\right)dt$$
$$-\frac{F_0}{1-\alpha}\int_0^T {}_tp_x\,\mu_{x+t}\int_{-\infty}^{\ln(c_t)} e^y\,g_S(y;(\mu-r-m)t;\sigma\sqrt{t};\omega)\,dy\,dt;$$

where $c_t = \left(e^{(\mu-r)t}G - V\right)/F_0$, and g_S and G_S are the pdf and cdf of the skew-normal distribution with location parameter $(\mu-r-m)t$, scale parameter $\sigma\sqrt{t}$, and shape parameter ω, respectively.

Proof. See Appendix A.4. □

3.2.4 Calculation notes for gross liabilities

Now, given that $\quad > \ _e$, the expression for V based on net liabilities of a GMMB rider, ${}_eL_n^0$, can be obtained from the following equation:

$$
\begin{aligned}
1 \quad &= {}_Tp_x \quad P \left[e^{\ rT}(G \quad F_T) \quad \int_0^T e^{\ rs} M_s ds > V \right] \\
&= {}_Tp_x \quad P \left[e^{\ (r+m)T} \frac{S_T}{S_0} + m_e \int_0^T e^{\ (r+m)s} \frac{S_s}{S_0} ds < \frac{e^{\ rT} G \quad V}{F_0} \right] \\
&= {}_Tp_x \quad P \left(T; \frac{e^{\ rT} G \quad V}{F_0} \right) ;
\end{aligned}
\qquad (3.28)
$$

where function P is given in (3.27) and at a significance level of .

Note that under the normal model, an explicit expression of $P(T;x)$ is presented in Feng and Volkmer (2012) (see Equation (3.5) in Proposition 3.3). However, when $S_T = S_0$ follows a Cauchy or skew-normal model, the explicit expression for $P(T;x)$ is not available. We propose the Monte Carlo simulation algorithm below for computing (i.e., approximating) $_e$ and V .

Let

$$
e^{Y_s} = e^{\ (r+m)s} \frac{S_s}{S_0}; \qquad 0 \quad s \quad T: \qquad (3.29)
$$

Note that the integral $\int_0^T e^{Y_s} ds$ is defined path by path. For a fixed sample path of $\{Y_s\}_{s \ 0}$, the integral $\int_0^T e^{Y_s} ds$ is a continuous function, so the integral can be calculated or approximated over the interval $[0;T]$. Assume a constant time increment of a unit with in total n units in a year (so that these are nT time units in T complete years), and then for a fixed sample path of $\{Y_s\}_{s \ 0}$, we have

$$
\int_0^T e^{Y_s} ds \quad \sum_{k=1}^{nT} e
$$

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Step 5: Compute the estimated V_α by

$$\hat{V}_\alpha = \frac{1}{M}\sum_{j=1}^{M} V_\alpha^{(j)}.$$

We now present the steps for calculating the conditional tail expectation CTE_α for the GMMB net liabilities. Given that $\tau > \tau_e$, and by Equation (3.3), the expression of CTE for net liabilities of GMMB, ${}_eL_n^0$, is given by

$$CTE_\alpha = \frac{{}_Tp_x}{1-\alpha} E\left[\left(e^{-rT}G - e^{-rT}F_T - \int_0^T e^{-rs}M_s ds\right) I_{\left\{e^{-rT}G - e^{-rT}F_T - \int_0^T e^{-rs}M_s ds > V_\alpha\right\}}\right];$$

at a significance level α.

By using (3.8) and the definition of e^{Y_s} given in (3.29), CTE_α can be written as

$$CTE_\alpha = \frac{{}_Tp_x}{1-\alpha} e^{-rT}G\, P\left[e^{Y_T} + m_e\int_0^T e^{Y_s}ds < \frac{e^{-rT}G - V_\alpha}{F_0}\right]$$
$$- \frac{{}_Tp_x}{1-\alpha} F_0\, E\left[\left(e^{Y_T} + m_e\int_0^T e^{Y_s}ds\right) I_{\left\{e^{Y_T} + m_e\int_0^T e^{Y_s}ds < \frac{e^{-rT}G - V_\alpha}{F_0}\right\}}\right].$$

We further let

$$Z(T;x) = E\left[\left(e^{Y_T} + m_e\int_0^T e^{Y_s}ds\right) I_{\left\{e^{Y_T} + m_e\int_0^T e^{Y_s}ds < x\right\}}\right]. \qquad (3.33)$$

Then, using (3.28), we can obtain the following expression for CTE_α:

$$CTE_\alpha = e^{-rT}G$$

and CTE can be approximated by

$$\mathrm{CTE} \quad e^{rT} G \quad {}_{nT}p_x \frac{F_0}{1} \bar{Z} \quad nT; \frac{e^{rT} G \quad V}{F_0} \; : \qquad (3.35)$$

Below are the detailed steps.

Step 1: Simulate N sets of $e^{Y_k} \,{}_{k=1}^{nT}$ and calculate corresponding N realizations of Q, $Q_1; Q_2; \ldots; Q_N$, based on the Cauchy model.

Step 2: Follow Steps 2–3 in **Algorithm 2** to obtain V

$$= \sum_{k=1}^{T} P\left[e^{-(r+m)k}\frac{S_k}{S_0} + m_d \int_0^k e^{-(r+m)s}\frac{S_s}{S_0}ds < \frac{e^{(\; - r)k}G}{F_0} \right] {}_{k-1}p_x q_{x+k-1}$$

$$= \sum_{k=1}^{T} P\left(\; k; \frac{e^{(\; - r)k}G}{F_0} \right) {}_{k-1}p_x q_{x+k-1};$$

where function P is given in Equation (3.27) and m_e is replaced by m_d. Then, $\;_d$ for the GMDB net liabilities can be expressed as

$$_d = 1 - \sum_{k=1}^{T} {}_{k-1}p_x q_{x+k-1} P\left(\; k; \frac{e^{(\; - r)k}G}{F_0} \right):$$

Similarly, given that $\; > \;_d$, V for net liabilities of GMDB, ${}_dL_n^0$, can be obtained from the following equation:

$$1 - \; = \sum_{k=1}^{T} {}_{k-1}p_x q_{x+k-1} P\left[e^{-rk}(e^{\; k} G - F_k) - \int_0^k e^{-rs} M_s ds > V \right]$$

$$= \sum_{k=1}^{T} {}_{k-1}p_x q_{x+k-1} P\left[e^{-(r+m)k}\frac{S_k}{S_0} + m_d \int_0^k e^{-(r+m)s}\frac{S_s}{S_0}ds < e_1 p \right.$$

Algorithm 4: approximate $_d$

Based on the approximation formula of Equation (3.37), $_d$ can be approximated by

$$_d \quad 1 \quad \sum_{k=1}^{T} {}_{k-1}p_x q_{x+k-1} \tilde{P} \left(nk; \frac{e^{(\quad r)k} G}{F_0} \right) : \tag{3.38}$$

Below are the detailed steps.

Step 1: Sim

Step 6: Compute the estimated V by

$$\hat{V} = \frac{1}{M} \sum_{j=1}^{M} V^{(j)}.$$

We now present the steps for calculating the conditional tail expectation CTE for the

Using the same approximation for $P(k;x)$ as in (3.37), CTE can be approximated by

$$\mathrm{CTE} \quad \frac{1}{1} \sum_{k=1}^{T} e^{(\quad r)k} G \quad \tilde{P} \left(nk; \frac{e^{(\quad r)k} G \quad V}{F_0} \right)$$
$$F_0 \quad \tilde{Z} \left(nk; \frac{e^{(\quad r)k} G \quad V}{F_0} \right)\Bigg) \quad {}_{k \quad 1}p_x \, q_{x+k \quad 1}: \qquad (3.41)$$

Below are the detailed steps.

Step 1: Simulate N sets of $\left\{ e^{Y_l} \right\}_{l=1}^{nT}$ and calculate corresponding N sets of realizations of Q_k^0, for $k = 1; 2; \ldots ; T$, denoted as $Q^0_{(k;1)}; \ldots ; Q^0_{(k;N)}$, based on the Cauchy model.

Step 2: Follow Steps 2–4 in **Algorithm 5** to obtain $V^{(1)}$.

Step 3: For each k, calculate $\tilde{P}\left(nk; \left(e^{(\quad r)k} G \quad V \right) = F_0 \right)$ using the value calculated in Step 3 in **Algorithm 5** and calculate the value of $\tilde{Z}\left(nk; \left(e^{(\quad r)k} G \quad V^{(1)} \right) = F_0 \right)$ using the following empirical formula:

$$\tilde{Z} \left(nk; \frac{e^{(\quad r)k} G \quad V^{(1)}}{F_0} \right) = \frac{1}{N} \sum_{l=1}^{N} Q^0_{(k;l)} I_{\left\{ Q^0_{(k;l)} < \frac{e^{(\quad r)k} G \quad V^{(1)}}{F_0} \right\}} ;$$

and then calculate $\mathrm{CTE}^{(1)}$ using (3.41).

Step 4: Repeat Steps 1–3 m 53.5e(F)]80 J 0.398 w 0 0 m 9.901 Tf 3.293 0 Td [(k)-27(;l)86 8.18266 8.18266

Chapter 4

Numerical illustrations

4.1 Data and Models

In this section, we first introduce the data and perform a preliminary data analysis by looking at the histograms, time series plots, and autocorrelation function plot of data. In Section 4.1.2, we present the maximum likelihood estimation method for estimating the model parameters for normal, Cauchy, and skew-normal models, and then determine the better-fit distributions based on some model selection criteria. In Section 4.1.3, we provide a graphical comparison of theoretical and the empirical distributions and examine simulated projections.

4.1.1 Data

In this project, we use the S&P 500 weekly stock index prices[1] over the past two decades, between the week of February 6^{th} 2000 and the week of January 26^{th} 2020, as our historical data. We calculate the returns by taking the logarithm of the ratio of two consecutive stock index prices. The time series plot of historical weekly stock index prices are shown in Figure 4.1.

The historical weekly returns are plotted in Figure 4.2 and the relevant statistics of this data are shown in Table 4.1. From Table 4.1, we see that the skewness and kurtosis of historical returns data are 0.8928359 and 10.42228, respectively. The empirical skewness of a data is a measure of asymmetry of the empirical distribution. The data is symmetrically distributed if its empirical skewness has a value of 0, and the empirical distribution is left-skewed (right-skewed) if its empirical skewness has a negative (positive) value. The empirical kurtosis of a data is a measure that assesses whether the data are heavy-tailed or light-tailed relative to a normal distribution. The data is normally distributed if its empirical excess kurtosis has a value of 0, and the data has heavier (lighter) tails than normal if its empirical excess kurtosis has a positive (negative) value. Based on the skewness and excess kurtosis

[1] https://ca.finance.yahoo.com/

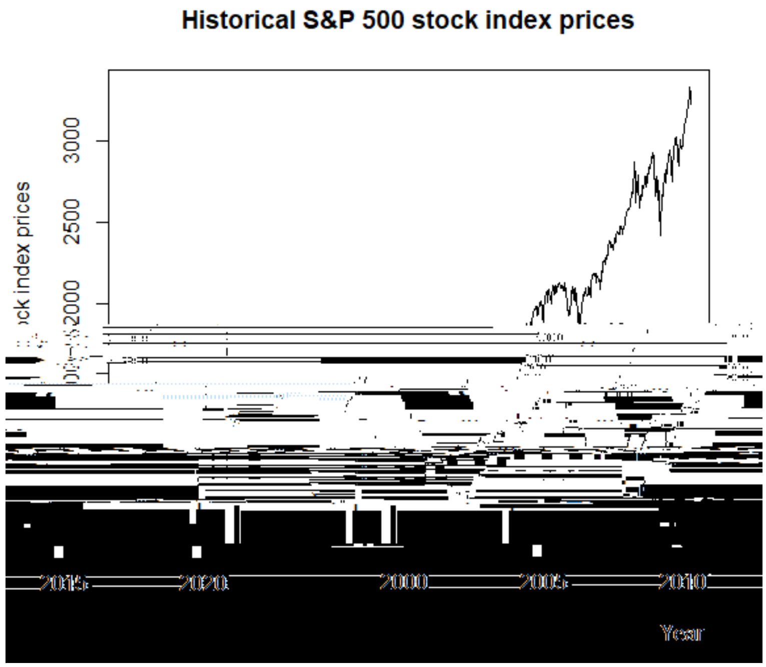

Figure 4.1: Time series plot of S&P 500 weekly prices.

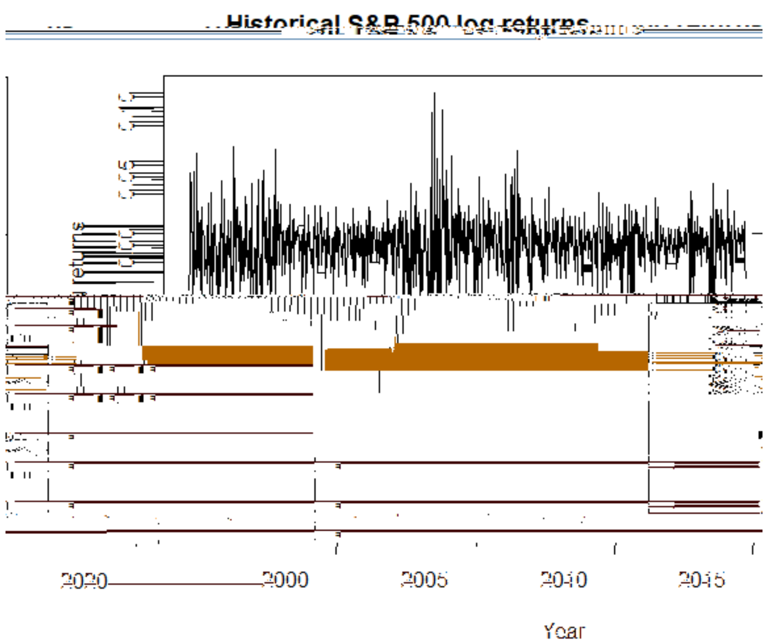

Figure 4.2: Time series plot of S&P 500 weekly returns.

obtained from our historical returns data, we can conclude that the empirical distribution of historical returns is left-skewed and has heavy tails. This can also be observed from the histogram of historical weekly returns based on the stock index prices shown in Figure 4.3.

Table 4.1: Statistics for historical S&P500 weekly returns.

Min	Max	Median	Mean	Variance	Skewness	Kurtosis
-0.2008375	0.113559	0.0020705	0.0008098	0.0005717	-0.8928359	10.42228

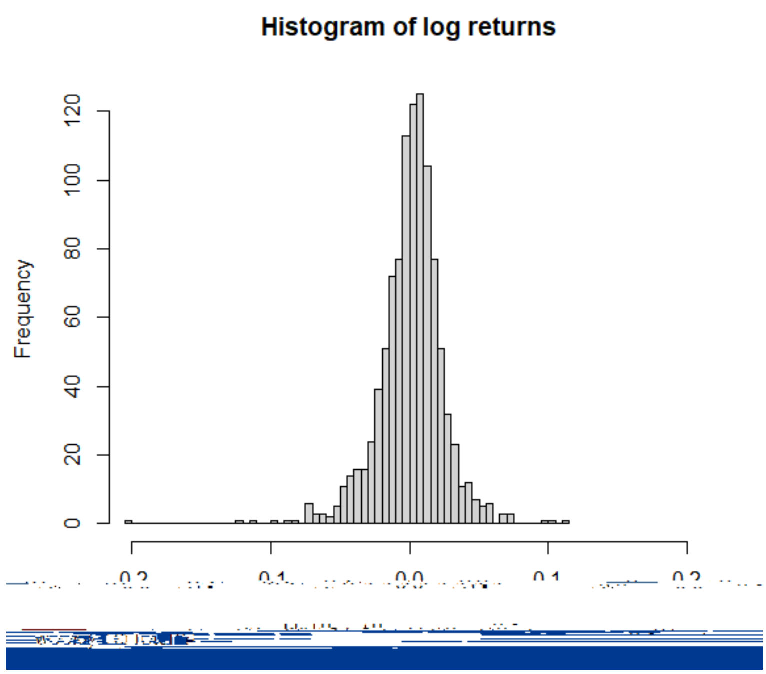

Figure 4.3: Histogram of S&P 500 weekly returns.

In this study, we assume that the returns are independent and identically distributed. To test the independence assumption of our returns data, we plot the autocorrelation function (ACF) of our historical weekly data in Figure 4.4. In time series studies, the autocorrelation function measures the correlation of a time series with itself after lagging. It can be observed from Figure 4.4 that the data have correlation of 1 at lag 0. This means that the data is perfectly correlated with respect to itself. The dashed blue lines represent a confidence interval of zero correlations. As we can see from Figure 4.4 and for any positive lag levels, the values of sample autocorrelation function are all within the dashed blue lines, implying that the historical lagged returns are not correlated.

In addition to the ACF plot, both Ljung-Box and Box-Pierce tests are also commonly used to verify the independence assumption of time series data. The Ljung-Box test, proposed by Ljung and Box (1978), examines whether a time series contains autocorrelation. The Box-Pierce test, proposed by Box and Pierce (1970), is a simplified version of Ljung-Box test. Both tests set up the null hypothesis in the same way; the null hypothesis assumes that the time series data are independently distributed. We perform both tests for our returns data at a lag level of 1 and their p values are obtained using the stats[2] package in R. The p values are 0.02677 and 0.02699, respectively. Based on these p values, we fail to reject the null hypothesis of both Ljung-Box and Box-Pierce tests at 1% significance level. Hence, the independence assumption should hold for our historical returns data.

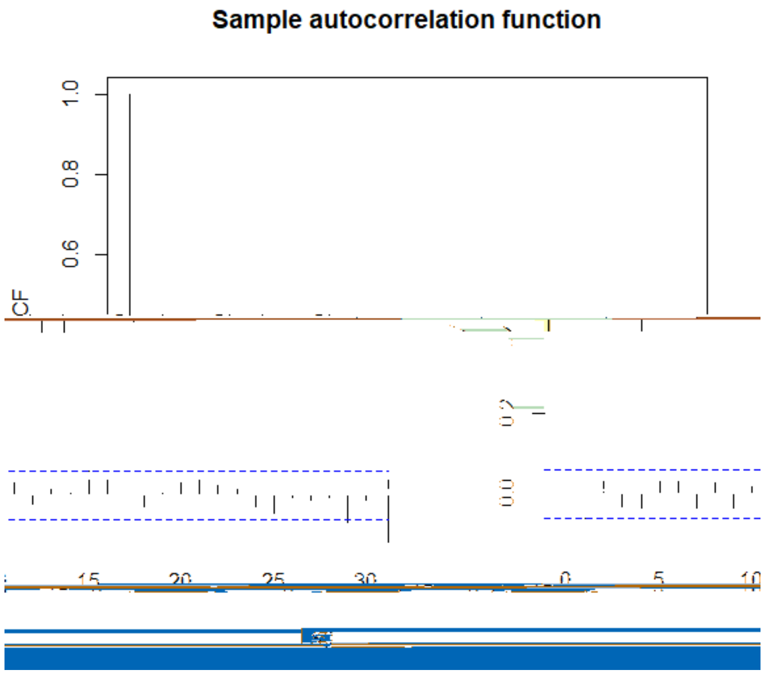

Figure 4.4: Sample ACF of S&P 500 weekly returns.

4.1.2 Models and estimations

Recall that the asset price at time t $(0 \leq t \leq T)$ is denoted as S_t, and the logarithm of the quotient of two consecutive stock prices at time $t-1$ and t, also known as equity return at time t, is denoted by $\ln(S_t/S_{t-1})$.

Let $X_i = \ln(S_i/S_{i-1})$, $i = 0,1,\ldots,n$. We first consider the lognormal model. In this case, the X_is are assumed to be independent and normally distributed with mean μ and standard deviation σ. The pdf and cdf of this normal distribution are given by (3.10) and (3.11), respectively.

In this project, we use the maximum likelihood estimation (MLE) method to estimate the parameters for the three distributions we consider. In the normal distribution case, the explicit expressions of the MLE of the parameters can be easily obtained by solving a

[2] https://www.rdocumentation.org/packages/stats/versions/3.6.2

system of equations, called estimating equations. The likelihood function based on a sample of observations $x_1, x_2, \ldots, x_n$, $L(\mu, \sigma^2; x_1, x_2, \ldots, x_n)$, is given by

$$L(\mu, \sigma^2; x_1, x_2, \ldots, x_n) \propto \frac{1}{\sigma^{2n}} \exp\left(-\frac{\sum_{i=1}^{n} (x_i - \mu)^2}{2\sigma^2} \right),$$

and the log-likelihood function can be expressed as

$$\ell(\mu, \sigma^2; x_1, x_2, \ldots, x_n) = -\frac{n}{2} \ln(\sigma^2) - \frac{\sum_{i=1}^{n} (x_i - \mu)^2}{2\sigma^2}.$$

By taking the first derivative of the log-likelihood function with respect to parameter μ and σ^2, respectively, and setting them equal to 0, we have the following system of estimating equations:

$$\begin{cases} \sigma^{-2} \left(\sum_{i=1}^{n} x_i - n\mu \right) = 0, \\ -\frac{n}{2\sigma^2} + \frac{\sum_{i=1}^{n} (x_i - \mu)^2}{2\sigma^4} = 0. \end{cases}$$

By solving this system of equations, we get

$$\begin{cases} \hat{\mu} = \frac{\sum_{i=1}^{n} x_i}{n}, \\ \hat{\sigma}^2 = \sum_{i=1}^{n} (x_i - \ldots \end{cases}$$

Table 4.2: The estimated parameters and maximum log-likelihood.

Model	Parameters			Log-likelihood
Normal	= 0.00080985	= 0.02390084		2412.129
Cauchy	= 0.00284392	= 0.01103033		2422.943
Skew-normal	= 0.00048935	= 0.02363083	! = -0.26741422	2431.782

Table 4.2 shows the values of the estimated parameters for the three models. We notice that both Cauchy and skew-normal models have larger log-likelihood values at the estimated parameters than that of the normal model case. Based on this criterion, both Cauchy and skew-normal models fit our historical data better than the normal one. To determine which model fits the data better out of models with different numbers of parameters, we use the Akaike information criterion (Akaike, 1973) and Bayesian information criterion (Schwarz, 1978).

Definition 4.1. The Akaike information criterion (AIC) is a measure of goodness of fit defined as

AIC = 2k 2`(^);

where k is the number of estimated parameters, ^ represents a set of estimated parameters in the model, and ` is the log-likelihood function.

Definition 4.2. The Bayesian information criterion (BIC) is a measure of goodness of fit defined as

BIC = k ln(n) 2`(^);

where n is the number of observations.

Both AIC and BIC can help to compare the goodness-of-fit for models with different numbers of parameters. A smaller AIC or BIC indicates a better-fit model within all the candidate models. We present the AIC and BIC values for the three models in Table 4.3.

Table 4.3: The AIC and BIC values of models.

Model	Number of parameters	Log-likelihood	AIC	BIC
Normal	2	2412.129	-4820.257	-4810.360
Cauchy	2	2422.943	-4841.885	-4831.987
Skew-normal	3	2431.782	**-4857.564**	**-4842.717**

According to the AIC and BIC values in Table 4.3, the skew-normal model has the lowest AIC and BIC values. Hence, we can conclude that the skew-normal distribution is the best-fit model for our S&P 500 historical returns data.

4.1.3 Graphical analysis of tted models

In this subsection, we use several graphical tools to help understand the relationship between the tted theoretical distributions and the empirical distribution. The useful plots include

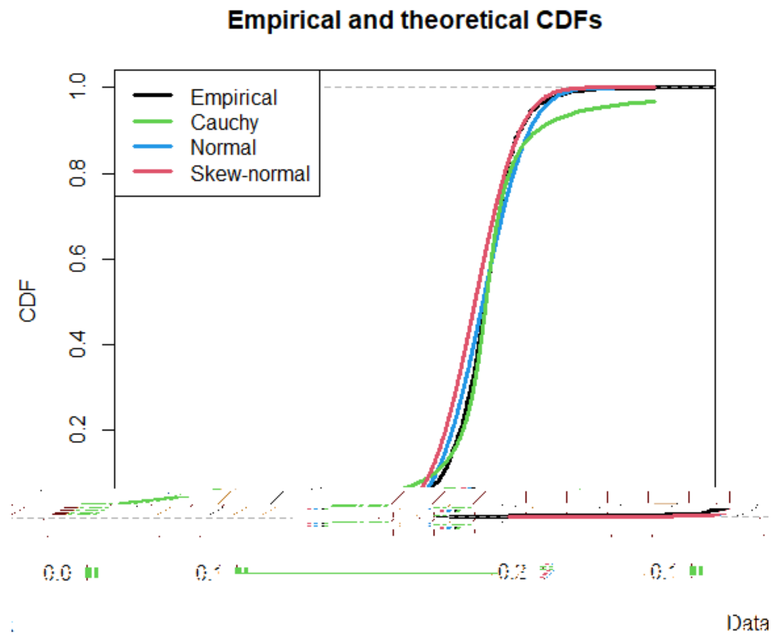

Figure 4.6: CDF plot of theoretical and empirical distributions.

In Figure 4.7, we show the quantiles of the fitted three distributions against the empirical distribution on the left, and the cumulative probabilities of the fitted three distributions against the empirical distribution on the right. We observe from both the Q-Q plots and P-P plots that the Cauchy distribution shows a good fit for the values around 0 in the middle part of the empirical distribution. The skew-normal Q-Q plot illustrates again that the empirical distribution is negatively skewed.

In Figure 4.8, we display S&P 500 projections for the next 10 years based on the fitted normal, Cauchy, and skew-normal models on the left, and corresponding return projections for the next 10 years on the right. Note that the lower and upper quantiles of price projections showed in the figure are 25% and 75% for the Cauchy model, and 5% and 95% for the normal and skew-normal models. The projections of the stock prices under the Cauchy model show an extremely wide price range compared to that under the normal and skew-normal models. Moreover, the extremes of the projected prices under the Cauchy model become more extreme to the upside. The reason for this phenomenon is that the Cauchy distribution is heavy-tailed. As expected, the prediction of returns under the Cauchy and normal models are symmetric, while that under the skew-normal model shows negative skewness. Because of the negative skewness of our fitted skew-normal model, the prediction shows a downward trend of stock prices. In addition, the median of projected returns is below 0 (under the green horizontal line) in the skew-normal model, which demonstrates the downward trend of its projected stock prices.

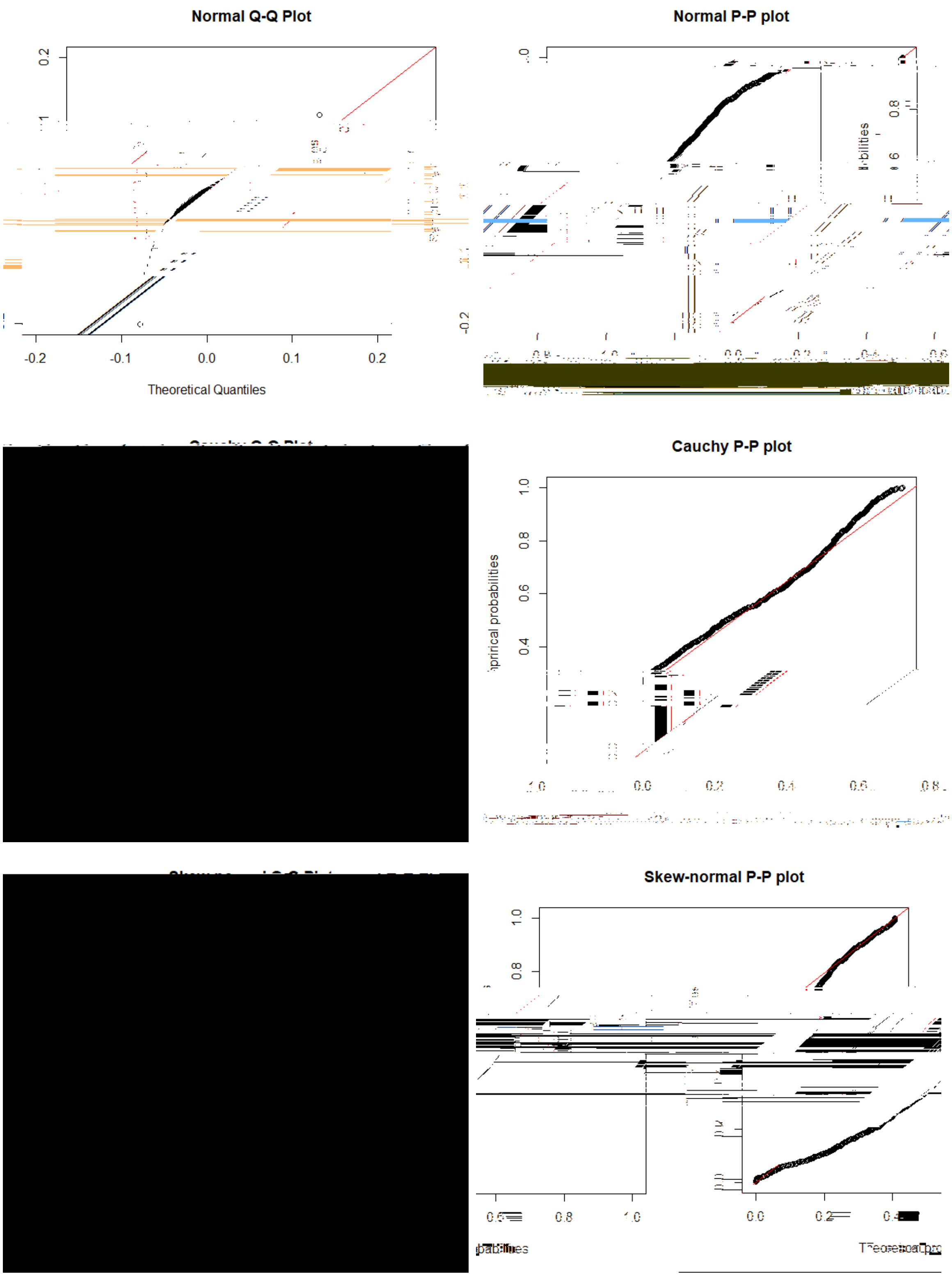

Figure 4.7: Q-Q plots and P-P plots.

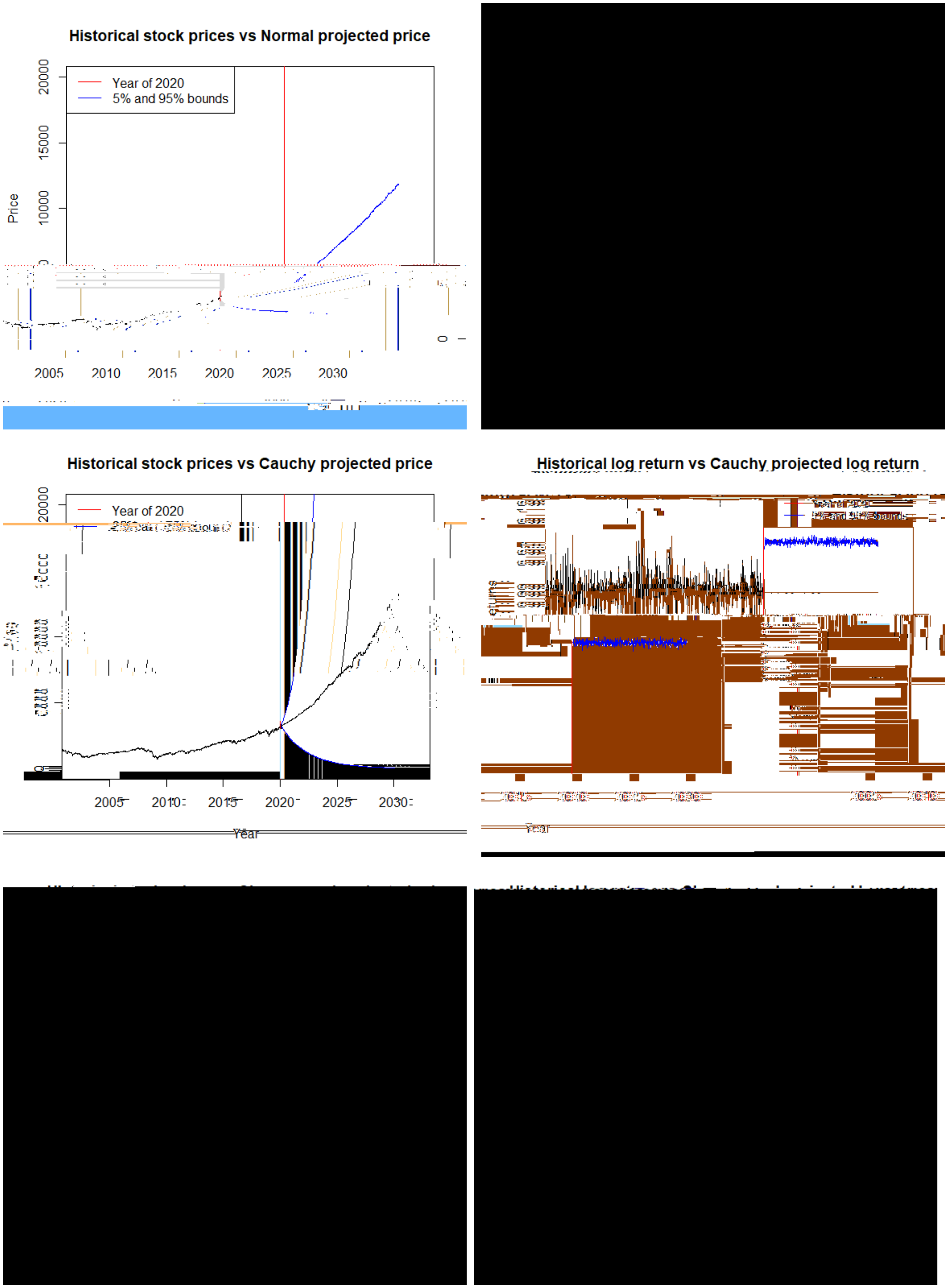

Figure 4.8: Projections versus historical data.

4.2 Risk measure results and analysis

In this section, we calculate and compare the VaR and CTE for the insurer's future liabilities (gross and net) with respect to GMMB and GMDB riders based on three fitted return

Table 4.5 shows the calculated risk measure results relative to the initial fund value F_0 for the GMMB gross liabilities with different predetermined guaranteed levels and different risk levels for normal, Cauchy, and skew-normal underlying equity models. From the insurer's points of view, only the positive liabilities are meaningful in real life applications. The values with an asterisk mark in Table 4.5 imply negative risk measures for gross liabilities. For example, with the guaranteed level of 75% and the normal model, the insurer has a risk capital of 0% of the initial fund, which indicates that no capital is exposed to risk at levels of 80% and 90%. In this case, such products with corresponding risk levels and guaranteed levels could be profitable.

Table 4.5: The numerical results of GMMB gross liabilities.

		Models		
Guaranteed level (%)	Risk Measures	Normal	Cauchy	Skew-normal
75	$V_{80\%}=F_0$	0*	0.48871	0.00056
	$CTE_{80\%}=F_0$	0.01617	0.50137	0.12680
	$V_{90\%}=F_0$	0*	0.50273	0.11422
	$CTE_{90\%}=F_0$	0.11071	0.50274	0.19775
	$V_{95\%}=F_0$	0.09601	0.50274	0.18644
	$CTE_{95\%}=F_0$	0.17584	0.50274	0.29438
100	$V_{80\%}=F_0$	0.01490	0.65629	0.16814
	$CTE_{80\%}=F_0$	0.18375	0.66895	0.29438
	$V_{90\%}=F_0$	0.16744	0.67031	0.28179
	$CTE_{90\%}=F_0$	0.27829	0.67032	0.36533
	$V_{95\%}=F_0$	0.26359	0.67032	0.35402
	$CTE_{95\%}=F_0$	0.34342	0.67032	0.41457
120	$V_{80\%}=F_0$	0.14896	0.79035	0.30221
	$CTE_{80\%}=F_0$	0.31781	0.80301	0.42844
	$V_{90\%}=F_0$	0.30150	0.80437	0.41586
	$CTE_{90\%}=F_0$	0.41235	0.80438	0.49940
	$V_{95\%}=F_0$	0.39765	0.80438	0.48808
	$CTE_{95\%}=F_0$	0.47749	0.80438	0.54863

Based on the values showed in Table 4.5, we notice that all the risk measures relative to the initial fund value F_0 for the Cauchy model are significantly greater than that of the normal and skew-normal models. For the guaranteed level at 120% of the initial premium, around 80% of the insurer's capital is exposed. This reminds that the Cauchy model should be used with a great caution. Because the Cauchy distribution features fat-tails compared to the normal and skew-normal distributions, it is more likely to incur enormous future losses if the Cauchy distribution is used.

Table 4.6 shows the calculated risk measure results relative to F_0 for GMMB net liabilities with different predetermined guaranteed levels and different risk levels for normal, Cauchy, and skew-normal models. Because the net liability is the gross liability net of the margin o set, we expect the risk measure values in Table 4.6 to be all slightly less than the corresponding ones showed in Table 4.5. However, this holds only for the normal and Cauchy models. By comparing the risk measure values in both Tables 4.5 and 4.6 for the skew-normal distribution, we notice that the net liability risk measure values are larger than those for the gross liability. This is because the fitted skew-normal distribution is negatively (left) skewed, which would cause the simulated future paths of underlying equity prices have a downward trend.

Table 4.6: The numerical results of GMMB net liabilities.

		Models		
Guaranteed level (%)	Risk Measures	Normal	Cauchy	Skew-normal
75	$V_{80\%}=F_0$	0*	0.39717	0.08511
	$CTE_{80\%}=F_0$	0*	0.47745	0.18810
	$V_{90\%}=F_0$	0*	0.48843	0.17772
	$CTE_{90\%}=F_0$	0.08851	0.49754	0.24529
	$V_{95\%}=F_0$	0.07433	0.49820	0.23647
	$CTE_{95\%}=F_0$	0.15524	0.50115	0.28471
100	$V_{80\%}=F_0$	0*	0.56594	0.25342
	$CTE_{80\%}=F_0$	0.15964	0.64497	0.35547
	$V_{90\%}=F_0$	0.14247	0.65607	0.34543
	$CTE_{90\%}=F_0$	0.25645	0.66513	0.41288
	$V_{95\%}=F_0$	0.24306	0.66580	0.40355
	$CTE_{95\%}=F_0$	0.32256	0.66874	0.45279
120	$V_{80\%}=F_0$	0.12151	0.70024	0.38716
	$CTE_{80\%}=F_0$	0.29329	0.77935	0.48946
	$V_{90\%}=F_0$	0.27647	0.79002	0.47897
	$CTE_{90\%}=F_0$	0.39037	0.79918	0.54687
	$V_{95\%}=F_0$	0.37543	0.79983	0.53804
	$CTE_{95\%}=F_0$	0.45694	0.80281	0.58747

4.2.3 Risk measure results for GMDB

The GMDB rider provides the policyholder a death benefit during the policy term. The death benefit is equal to the larger value between the guaranteed amount accumulated at a roll-up rate and the separate account fund value at the time of death of the policyholder. In this numerical example, we assume that the death benefit is payable at the end of the year of death, and we calculate the risk measures for the insurer's future gross liabilities based on the propositions presented in Section 3.2.3, and the risk measures for the net liabilities based on the algorithms presented in Section 3.3.2. Below is the valuation basis used for calculations for the GMDB rider:

^

From the insurer's point of view, the risk measures for net liabilities are more meaningful than that for gross liabilities. Table 4.8 shows the calculated risk measure values relative to F_0 for net GMDB liabilities. The values calculated for net GMDB liabilities are smaller than those for the gross GMDB liabilities for the normal and Cauchy models only. We also observe that, with either a GMMB rider or a GMDB rider, the Cauchy model returns the largest risk measure relative to the initial fund value F_0 due to the fat tails of the Cauchy distribution, while the skew-normal model returns larger results than normal model. The latter is because the estimated location parameter of the skew-normal model is smaller than the estimated location parameter of the normal model. In addition, the estimated negative shape parameter corresponding to the negative (left) skewness for the skew-normal model implies a fat left tail in distribution compared to the normal model, which results in larger risk measure values in this case.

Table 4.8: The numerical results of GMDB net liabilities.

		Models		
Guaranteed level (%)	Risk Measures	Normal	Cauchy	Skew-normal
75	$V_{80\%}=F_0$	0*	0*	0*
	$CTE_{80\%}=F_0$	0.06966	0.26435	0.16508
	$V_{90\%}=F_0$	0*	0.00753	0.13942
	$CTE_{90\%}=F_0$	0.13890	0.53101	0.29654
	$V_{95\%}=F_0$	0.11369	0.63524	0.28510
	$CTE_{95\%}=F_0$	0.24150	0.68515	0.38128
100	$V_{80\%}=F_0$	0*	0*	0.03587
	$CTE_{80\%}=F_0$	0.20713	0.38692	0.37423
	$V_{90\%}=F_0$	0.17880	0.16539	0.37521
	$CTE_{90\%}=F_0$	0.36872	0.76163	0.52977
	$V_{95\%}=F_0$	0.34861	0.86678	0.51769
	$CTE_{95\%}=F_0$	0.47483	0.91904	0.61174
120	$V_{80\%}=F_0$	0*	0*	0.22600
	$CTE_{80\%}=F_0$	0.37253	0.48659	0.56205
	$V_{90\%}=F_0$	0.36891	0.36146	0.56039
	$CTE_{90\%}=F_0$	0.55630	0.95021	0.71322
	$V_{95\%}=F_0$	0.53731		

Chapter 5

Conclusion

In this project, we studied variable annuities with two types of guaranteed benefits: the GMMB and GMDB riders. We assumed that the returns of the underlying asset for the considered variable annuities follow Cauchy or skew-normal distributions. Two typical risk measures, the VaR and the CTE, are calculated for the insurer's future gross and net liabilities with either of the two guaranteed benefit riders. In an illustration, we fitted our proposed asset return models to the historical S&P 500 weekly returns data. We then compared the calculated risk measure results under the fitted asset return models for both insurer's gross and net liabilities with one of the guaranteed benefit riders.

Our main findings of this study are as follows. First, we found that our newly proposed Cauchy and skew-normal models can fit the returns data better than the normal model under the maximum likelihood estimation. While the Cauchy model can capture the peak of the empirical distribution of the returns, the skew-normal model is suitable for left or right ity contracts. Insurers need to be aware of the positive and negative aspects of using the distributions we study in this project.

This study can be extended in different ways. We could consider other distributional models (for example, the skew t distribution) to model the underlying asset returns, and we may also consider mixture models which are able to capture the peak, skewness, and heavy tails of the equity returns. We may also consider a variable annuity with both GMMB and GMDB riders and study the risk measures of the insurer's liabilities in this case. In addition, we may use alternative risk measures to calculate the risk capitals of insurers' future liabilities. For example, the weighted value-at-risk proposed by Cont et al. (2010), also called the range value-at-risk (RVaR), is the truncation version of the CTE. The RVaR is suitable in dealing with the fat-tail distributions and infinite tail expectations (Bairakdar et al., 2020).

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Appendix A

Proofs

The proof for the results presented in Propositions 3.1 to 3.4 for the normal model are given in Feng and Volkmer (2012). Here, we prove only the results for the Cauchy model. (i.e., case (2) in each of Propositions 3.1 to 3.4). The results under the skew-normal model can be proven similarly, so they are omitted.

A.1 Proof of Proposition 3.1

Since $S_T = S_0$ Log-Cauchy(T; T), we can easily get that

$$\frac{S_T}{S_0} e^{(r+m)T} \quad \text{Log-Cauchy}((\quad r \quad m)T;\; T\,);$$

or

$$\ln \frac{S_T}{S_0} e^{(r+m)T} \quad \text{Cauchy}((\quad r \quad m)T;\; T\,):$$

Let $= (1- \;)/{}_T p_x$. Then, we have

$$\ln \left(\frac{e^{rT} G \quad V}{F_0} \right)^{!} = (\quad r \quad m)T + \; T c \;;$$

where c is the 100 % percentile of the standard Cauchy distribution. This gives

$$V = e^{rT} G \quad F_0 \exp f(\quad r \quad m)T + \; T c \; g;$$

which proves (3.20). □

A.2 Proof of Proposition 3.2

Proof. From Equation (3.7) and because the future lifetime of the policyholder is independent of F_T, the CTE for gross liability of a GMMB rider is given by

$$CTE = \frac{{}_T p_x}{1} E^{h} e^{rT} (G \quad F_T) I_{f e^{rT} (G \quad F_T) > V \; g}{}^{i};$$

when $> \; _e$.

Using (3.8) and letting $Y = \ln(S_T = S_0 \;\; e^{(r+m)T})$ and $a = (e^{rT} G \quad V\,) = F_0$, we have

$$CTE = \frac{{}_T p_x}{1} E^{2}4 \;\; e^{rT} G \quad F_0 \frac{S_T}{S_0} e^{(r+m)T} \quad I^{n}{}_{e^{rT} G \quad F_0 \frac{S_T}{S_0} e^{(r}} \quad = \quad {}^{S_x}_{T}$$

A.3 Proof of Proposition 3.3

Proof. Recall that the gross liability of a GMDB rider defined by (3.4) is

$${}_dL_g^0 = e^{\ r\ x}(e^{\ x}G \quad F_{\ x})_+ I(\ x \quad T):$$

From Equation (3.6) and by conditioning on the future lifetime of the policyholder $_x$, V can be determined by

$$\begin{aligned} 1 \quad &= \int_0^T P\left[{}_dL_g^0 > V\right] f_{\ x}(t)dt; \\ &= \int_0^T P\left[e^{\ rt} \quad e^{\ t}G \quad F_t \quad > V\right] f_{\ x}(t)dt; \end{aligned}$$

when $\ > \ _d$, and where $f_{\ x}(t) = {}_tp_x \quad _{x+t}$.

Using (3.8), we have

$$\begin{aligned} 1 \quad &= \int_0^T P\left(e^{\ rt} \quad e^{\ t}G \quad F_0\frac{S_t}{S_0}e^{\ mt} \quad > V\right) {}_tp_x \quad _{x+t}dt \\ &= \int_0^T P\left[\frac{S_t}{S_0}e^{\ (r+m)t} < \frac{e^{(\quad r)t}G \quad V}{F_0}\right] {}_tp_x \quad _{x+t}dt: \end{aligned}$$

Because $S_t = S_0 e^{\ (r+m)t}$ Log-Cauchy$((\quad r \quad m)t;\ t\ )$, V can then be determined by

$$1 \quad = \int_0^T F_C\left(\ln\left(\frac{e^{(\quad r)t}G \quad V}{F_0}\right);(\quad r \quad m)t;\ t\right) {}_tp_x \quad _{x+t}dt;$$

where F_C is the cdf of the Cauchy distribution. This proves (3.25). □

A.4 Proof of Proposition 3.4

Proof. From Equation (3.7) and by conditioning on the future lifetime of policyholder $_x$, the CTE for gross liability of a GMDB rider is given by

$$\begin{aligned} CTE \quad &= E\left[{}_dL_g^0 | {}_dL_g^0 > V\right] \\ &= \frac{1}{1}\int_0^T E\left[e^{\ rt} \quad e^{\ t}G \quad F_t \quad I_{\{e^{\ rt}(e^{\ t}G \quad F_t) > V\}}\right] {}_tp_x \quad _{x+t}dt; \end{aligned}$$

when $\ > \ _d$.

Using (3.8), and letting $Y_t = \ln(\ S_t = S_0 \quad e^{\ (r+m)t})$ and $c_t = (\ e^{(\quad r)t}G \quad V\ ) = F_0)$, we have

$$CTE \ = \frac{1}{1}\int_0^T E\left[e^{(\quad r)t}G \quad F_0\frac{S_t}{S_0}e^{\ (r+m)t} \quad I_{\{e^{(\quad r)t}G \quad F_0 \frac{S_t}{}\right.$$

Because Y_t