Nearly Orthogonal Arrays of Strength Three

by

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Abstract

Dedication

To my dearest husband, Viraj for his unconditional support, encouragement and motivation.

Acknowledgements

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of factors, which renders full factorials impractical. Fractional factorial designs provide a remedy for the expensive nature of full factorial designs. As the name itself suggests, such a design contains only a fraction of the complete set of runs required by its underlying full factorial design. yial design.

our empirical study. We provide a summary and conclusions in Chapter 4.

1.2 Fractional Factorial Designs

Fractional factorial designs are useful in examining the effects of a large number of factors on a response variable of interest using a relatively small number of experimental runs. Any experimental design that contains less than 2^m runs in an *m*-factor experiment is a fractional factorial. Regular fractional factorial designs are referred to as 2^{m-p} designs, where the *p* indicates the corresponding fraction of runs.

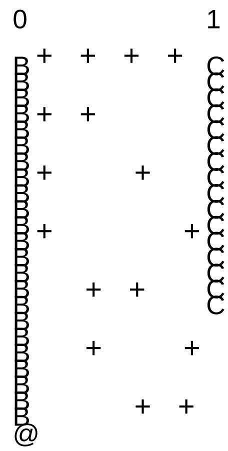
Consider a five-factor design each with two levels. A full factorial design requires $2^5 = 32$ runs to estimate all its effects as mentioned below.

		Interactions			
Average	Main Effects	2-factor	3-factor	4-factor	5-factor
1	5	10	10	5	1

Table 1.1: Main Effects and Interaction Effects of a Five-Factor Experiment

However, many of these effects might be negligible in practice. In general, higher order interactions are not of our interest. Very often, experimenters' main interest lies in estimating main effects, and hence, it is meaningless to carry out all 32 runs to estimate the thirty-one effects in the above example. Moreover, when an experiment involves a large number of factors, higher order interactions are often negligible. That's where fractional factorial designs come into play. For instance, a half fractional factorial design of a five-factor two-level experiment is a 2^{5-1} design. This design uses only sixteen runs to study five factors. Nevertheless, the significant results obtained from both designs will probably be similar (Dey and Mukerjee (1999)).

Regular fractional factorials are constructed using generators. For instance, in the above five-factor half fractional factorial experiment, a full 2^4 design is written for the first four



Orthogonal array with strength three. Any two columns in a

1.3 J-Characteristic and a Measure of Three-Dimensional Projection Property

J-characteristics are useful indicators to recognize the unique features of factorial designs. The J-characteristic of a single column x is expressed as $J(x)$, which gives the sum of the elements in that particular vector. The J-characteristic $J(x_1, x_2)$ represents the inner product of the two vectors, x_1 and x_2. All the properties of OAs are also fully determined by their J-characteristics. If all the J-characteristics of a factorial design for all possible subsets with t or less columns are zero, the design is an OA with strength t. Consider the design given in the above matrix with eight runs and four factors. If its columns are denoted by x_i, where i=1, 2, 3 and 4, we obtain that

$$J_1 = \{J(x_1), J(x_2), J(x_3), J(x_4)\} = \{0, 0, 0, 0\},$$

$$J_2 = \{J(x_1, x_2), J(x_1, x_3), J(x_2, x_3), J(x_1, x_4), J(x_2, x_4), J(x_3, x_4)\} = \{0, 0, 0, 0, 0, 0\},$$

$$J_3 = \{J(x_1, x_2, x_3), J(x_1, x_2, x_4), J(x_1, x_3, x_4), J(x_2, x_3, x_4)\} = \{0, 0, 0, 0\},$$

an expression for the three dimensional projection property for a factorial design D, as given below:

$$V_3(D) = 2^{-6} \sum_{j=1}^{3} \binom{m-j}{3-j} \sum_{|t|=j} J_t^2; \qquad (1.1)$$

where m is the total number of factors of the design D, and $|t|$ denotes the number of columns in a subset t of columns. Let

$$J_1\text{-Component} = 2^{-6} \binom{m-1}{2} \sum_{|t|=1} J_t^2,$$
$$J_2\text{-Component} = 2^{-6} \binom{m-2}{1} \sum_{|t|=2} J_t^2,$$
$$J_3\text{-Component} = 2^{-6} \sum_{|t|=3} J_t^2.$$

Equation (1.1) clearly explains that the V_3 is an overall measurement of J_1-, J_2- and J_3-components of a design. As discussed in Section 1.4, J_1 and J_2 characteristics of OAs of strength two are zero, and hence its V_3 measurement completely depends on the J_3-component. On the other hand the V_3 measurement of BFDs in our study are determined by considering only the J_2-component, as the corresponding J_1 and J_3 components are zero in fold-over designs as explained in the previous section. In general, a higher J_2 gives rise to a high variance whereas the bias of a design increases with the J_3. This relationship clearly explains why the OAs of strength two are variance-optimal and BFDs are bias-optimal. In conclusion, designs with the lowest V_3 are considered to be the best.

1.4 Bias, Variance, and Mean Squared Error

Our goal is to introduce a class of optimal designs with respect to some statistical criteria. We consider the concept of A-optimality in our study, which minimizes the sum of the variances of estimated main effects.

Still our main focus lies on estimating $\beta^{(1)}$ as two-factor interactions are not of our interest. However, $\hat{\beta}^{(1)}$ is no longer unbiased under the model indicated in equation (1.3) as proved below:

$$\begin{aligned} E(\hat{\beta}^{(1)}) &= (X_{(1)}^{T}X_{(1)})^{-1}X_{(1)}^{T}E(Y) \\ &= (X_{(1)}^{T}X_{(1)})^{-1}X_{(1)}^{T}(X_{(1)}\beta^{(1)} + X_{(2)}\beta^{(2)}) \\ &= \beta^{(1)} + (X_{(1)}^{T}X_{(1)})^{-1}X_{(1)}^{T}X_{(2)}\beta^{(2)}; \end{aligned}$$

$$\begin{aligned} Bias(\hat{\beta}^{(1)}; \beta^{(1)}) &= E(\hat{\beta}^{(1)}) - \beta^{(1)} \\ &= (X_{(1)}^{T}X_{(1)})^{-1}X_{(1)}^{T}X_{(2)}\beta^{(2)}. \end{aligned}$$

Nevertheless, the variance-covariance matrix of $\hat{\beta}^{(1)}$ is not affected by the two-factor interactions and hence remains unchanged in both models. However, as the estimator is biased under the new model, the MSE will be changed accordingly. The design which gives the minimum MSE is considered the best out of the set of designs being compared, where

$$MSE = \sigma^2 trace(X_{(1)}^{T}X_{(1)})^{-1} + ||(X_{(1)}^{T}X_{(1)})^{-1}X_{(1)}^{T}X_{(2)}\beta^{(2)}||^2. \quad (1.4)$$

All the above models take the intercept into consideration, while our main focus is to estimate main effects. Let $Var(\hat{\beta}^{(1)}) = \sigma^2 trace(M)$ with M being the matrix obtained by deleting the first row and the first column of $(X_{(1)}^{T}X_{(1)})^{-1}$, and $Bias(\hat{\beta}^{(1)}; \beta^{(1)}) = B\beta^{(2)}$ with B being the matrix obtained by deleting the first row of $(X_{(1)}^{T}X_{(1)})^{-1}X_{(1)}^{T}X_{(2)}$. Now under the model (1.3), the MSE for the main effects becomes:

$$\begin{aligned} MSE &= Var(\hat{\beta}^{(1)}) + ||Bias(\hat{\beta}^{(1)}; \beta^{(1)})||^2 \quad (1.5) \\ &= \sigma^2 trace(M) + ||B\beta^{(2)}||^2. \quad (1.6) \end{aligned}$$

Chapter 2

Design Methodology

In this chapter, we discuss three classes of designs used in our study. As mentioned before, our main goal is to give an alternative class of designs to use in the situations where OAs of strength three do not exist. Section 2.1 describes variance-optimal designs used in our study, whereas bias-optimal designs are given in Section 2.2. We introduce an alternative class of designs called nearly orthogonal arrays of strength three in Section 2.3 and discuss different approaches used to construct them. Finally, the method used to compare the three types of designs will be discussed in Section 2.4.

2.1 Variance-Optimal Designs

In this study, we consider designs with run sizes 10, 12, 14, 18, 20 and 28. For run sizes 12, 20 and 28, OAs of strength two are available. The OAs used in this study were constructed from Hadamard matrices. A Hadamard matrix is a square matrix with entries $+1$ and -1, whose rows are mutually orthogonal. This was initially introduced by Sylvester (1867) and later considered by Hadamard (1893). It has the mathematical property that $H^T H = HH^T = nI$, where I is an identity matrix with order n. Hadamard matrices are available for orders 1, 2 and for orders that are multiples of four. The method of tensor product allows the construction of Hadamard matrices with large orders from those with smaller orders. For example, a 4×4 Hadamard matrix can be constructed by a 2×2 Hadarmard matrix as illustrated below.

$$\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} & 1 \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \\ 1 \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} & 1 \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}$$

A Hadamard matrix is a valuable tool to construct OAs of strength two and three. Consider a Hadamard matrix with the first column containing only 1's, which is said to be normalized. Strength two OAs can be obtained by deleting the all 1's column of the corresponding Hadamard matrix. The resulting OA is denoted by OA$(n; 2^{n\ \ 1}; 2)$, which contains the maximum number of factors to be included in a strength-two OA with n runs. The array obtained by removing one or more columns is still orthogonal with strength two. Hence, a Hadamard matrix with order n can easily be used to construct an OA$(n; 2^{n^0}; 2)$ for any $n^0 \quad n \quad 1$.

We choose OAs of strength two with minimum G_2-aberration to be the variance-optimal designs, as they provide the minimum bias among all the non-isomorphic OAs of strength two, while minimizing the variance of estimated main effects among all possible designs. Let $s = \{x_1, \ldots, x_k\}$ represent any k subset of the design matrix D. Let

$$B_k(D) = \sum_{|s|=k} (J_k(s)/n)^2,$$

According to Tang and Deng (1999) and Schoen et al. (2017), for given designs D_1 and D_2, D_1 is said to have less G_2-aberration than D_2 if $B_r(D_1) < B_r(D_2)$, where r is the smallest integer such that $B_r(D_1)$ D_2

We also consider run sizes 10, 14 and 18 in our study, which involve non-orthogonal fractional factorial designs. According to Dey and Mukerjee (1999), when the run size of a design is even but not a multiple of four, the variance-optimal designs can be obtained by adding two specific runs to the closest lower order OA.

Lemma 1. The design obtained by adding two runs of form $l_1 = (1, \dots, 1)$ and $l_2 = (1, \dots, 1, -1, \dots, -1)$ to an orthogonal array is universally optimal and hence A-optimal.

Suppose a factorial design consists of m factors. Then l_1 is a vector containing m +1's and l_2 contains m_1 and m_2 +1's and -1's respectively, whereas m_1 is the largest integer that is smaller than or equal to $m/2$ and $m_2 = m - m_1$. To illustrate, we obtain a variance-optimal design for 10 runs with 5 factors by using five columns of the Hadamard matrix of order 8 and then adding two specific runs with l_1 being (1,1,1,1,1) and l_2 being (1,1,-1,-1,-1) to it. In short, while we use OAs of strength two for run sizes 12, 20 and 28, Lemma 1 is used for run sizes 10, 14 and 18 in order to obtain the variance-optimal designs. The below 10×5 variance-optimal design is obtained by considering the first five columns starting from the second column of the Hadamard matrix of order eight.

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ \hdashline 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{pmatrix}$$

2.2 Bias-Optimal Designs

Even though variance-optimal designs have the minimum variance, their MSE may get too large due to the bias. In our study, we also consider a special class of designs called bias-optimal designs, which provide zero bias and hence, their MSE is completely based on the variance portion. Margolin (1969) demonstrates that folding over an efficient non-orthogonal resolution III design with n runs produces a non-orthogonal resolution IV design with $2n$ runs. Such a design has zero bias and provides the minimum variance among all non-orthogonal fold-over designs. For convenience, they are called best fold-over designs (BFDs) throughout our study. In folding over a design, all the factor levels are reversed to form runs that are mirror images of those in the initial factorial design.

Nevertheless, BFDs do exist up to n=2 factors for a design with n runs. More precisely, when the run size is 10, a BFD is available for up to five factors. We use the efficient non-orthogonal resolution III designs summarized by Margolin (1969) to produce the BFDs of run sizes 10, 12, 14, 18, 20 and 28 with the number of factors being 5, 6, 7, 9, 10 and 14 respectively. The BFD for n = 10 and m = 5 is constructed as below, where the second half of the design is a mirror image of the first half.

$$
\begin{pmatrix}
1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 \\
\hdashline
1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1
\end{pmatrix}
$$

2.3 Nearly Orthogonal Arrays of Strength Three and Their Construction

In this section, we search for a set of alternative designs to use when OAs of strength three do not exist. Variance-optimal designs are efficient in optimizing the variance but their MSE may be too large due to the large bias. On the other hand, BFDs are bias-optimal, but the possible high variance may lead to a large MSE. We aim at constructing an alternative design, for each run size being considered in this study, to minimize both variance and bias to some extent. The design that has the lowest V_3 is chosen to be the best alternative design to use in our study. Such designs are similar to the OAs of strength three, as they are intended to minimize both bias and variance simultaneously in contrast to variance and bias-optimal designs. We call such factorial designs nearly orthogonal arrays (NOAs) of strength three.

We consider the construction of nearly orthogonal arrays of strength three under two scenarios. In the first scenario, we consider the situation of $m = n$=2, where both variance-optimal designs and BFDs are available. In the second scenario, we consider the number of factors in a design to be greater than the half of the number of runs by 1 ($m = n$=2 + 1), in which case BFDs are not available. The construction of NOAs of strength three is done separately under the two scenarios using two approaches, partially folding-over OAs of strength two and adding runs to OAs of strength two. Once the best design is found from the two approaches based on their V_3 values, a local search algorithm is used to improve it further. Those concepts are discussed in Subsections 2.3.1, 2.3.2 and 2.3.3, respectively.

2.3.1 Partial Fold-Over of OAs of Strength Two

In this3

NOA_0

Design	NOA_0	NOA_1	NOA_2	NOA_3	NOA_4	Best NOA
V_3	50	48	46.5	46	45	45

Table 2.1: Complete Set of Iterations of the Local Search Algorithm For the 20 10 Experiment

2.4 An MSE Criterion for Comparing Different Designs

The three dimensional projection property (V_3) provides a goodness measure for a fractional factorial design. However, it is combinatorial. We need to carry out a direct statistical comparison of the candidate designs to provide a clear guideline for practitioners to use in order where $\pi = \frac{N}{\binom{m}{2}}$ denotes the proportion of significant two-factor interactions, which is called the fraction of sparsity, $K_2 = tr(BB^T)$, and M and B matrices are previously defined in Section 1.3.

Chapter 3

Results of Comparisons

In this chapter, we summarize the important results of our study. We first describe how variance-optimal designs and bias-optimal designs perform using an example. The MSE is computed for both existing designs in 10 runs and 5 factors to illustrate their general behavior. In Section 3.1, we consider six experiments with 10, 12, 14, 18, 20 and 28 runs in $m = n$=2 factors. We conduct the same comparison for variance-optimal designs, bias-optimal designs and best NOAs of strength three in those experiments. The MSE calculation is done under seven different values of τ=σ, which is indicated by C throughout our study. For convenience, we take $\sigma = 1$ in all our calculations with $C = \tau$=$\sigma = \tau = 0.025$, 0.05, 0.1, 0.25, 0.5, 1 and 2. In every experiment, we also consider six different scenarios by changing the proportion of non-negligible two-factor interactions from 1 to 1=32. Thus, we take π = 1, 1/2, 1/4, 1/8, 1/16 and 1/32. Our goal here is to provide a range of C given by C_1 and C_2 such that the alternative design outperforms the corresponding two existing designs within the range of C_1 to C_2. In Section 3.2, we consider another six experiments having the same set of run sizes, but with $m = n$=2 + 1 factors, where the bias-optimal designs do not exist. Thus, we compare the available variance-optimal designs with the best NOAs for the same set of run sizes with 6, 7, 8, 10, 11 and 15 factors, respectively, by considering their MSEs. We provide a cut-off value C for C in each of the experiments so that the alternative design outperforms the corresponding variance-optimal design when $C > C$.

To understand the behaviour of variance-optimal designs and bias-optimal designs, we first examine in detail an example in which 5 factors are studied using 10 runs. We compute the MSE by taking effect sparsity into consideration as expressed in equation (2.1). Even though the variance of variance-optimal designs remains constant, the bias increases with the fraction of sparsity, which leads to the gradual increment in the MSE of estimated main effects. However, the MSE of bias-optimal designs remains unchanged as only the variance of such designs contributes to the MSE. This disproportionate behavior allows us to obtain a cut-off point, beyond which the best fold-over design performs better than the variance-optimal design of the same order. The example below summarizes the bias, variance and the MSE of estimated main effects of the two designs in an experiment with 10 runs and 5 factors when all two-factor interactions are significant.

n	m	$C = \tau = \sigma$	VOD			BFD		
			Variance	Bias	MSE	Variance	Bias	MSE
10	5	0.025	0.536	0.004	0.54	0.556	0	0.556
		0.05	0.536	0.014	0.55	0.556	0	0.556
		0.1	0.536	0.057	0.593	0.556	0	0.556
		0.25	0.536	0.353	0.889	0.556	0	0.556
		0.5	0.536	1.413	1.949	0.556	0	0.556
		1	0.536	5.653	6.189	0.556	0	0.556
		2	0.536	22.612	23.148	0.556	0	0.556

Table 3.1: Variance, Bias and MSE for Variance and Bias-Optimal Designs with 10 Runs and 5 Factors When All Two-Factor Interactions are Significant ($\pi = 1$)

Figure 3.1: Comparison of the MSE for Variance and Bias-Optimal Designs with n=10 and m=5 when = 1

Table 3.1 indicates that the MSE of the variance-optimal design is smaller at lower values of C, but gradually increases with to exceed the MSE of the bias-optimal design at a certain point, which is approximately 0.059 according to Figure 3.1. Hence, it is clear that the variance-optimal design is better at lower values of C, while the bias-optimal design outperforms the variance-optimal design when C exceeds 0.059. We can observe a similar trend for the two types of designs for other run sizes.

3.1 Comparison of Variance-Optimal Designs and Bias-Optimal Designs with Best Nearly Orthogonal Arrays of Strength Three

In this section, we carry out the same comparison by using the best NOAs of strength three. We choose designs with the lowest V_3 to be the best NOAs as discussed in Section 2.3. We consider six experiments having 10, 12, 14, 18, 20, 28 runs with 5, 6, 7, 9, 10, 14 factors respectively. In each experiment, we aim to compare all three types of designs together to identify three separate regions of C, in each of which one design performs better than the other two. The V_3 values and the corresponding J-components of the three types of designs in each experiment are displayed in Table 3.2. There are two best NOAs that perform equally well for run size 28. Hence, both of the designs are listed, as NOA 1 and NOA 2.

n	m	Design	V_3	J_1-Component	J_2-Component	J_3-Component
10	5	VOD	4.375	0.75	0.75	2.875
		BFD	1.875	0	1.875	0
		NOA	3.125	1.125	0.75	1.25
12	6	VOD	5	0	0	5
		BFD	6	0	6	0
		NOA	5	0	3	2
14	7	VOD	14.312	2.812	2.812	8.688
		BFD	59.063	0	59.063	0
		NOA	6.813	2.813	2.813	1.188
18	9	VOD	44.75	7	7	30.75
		BFD	29.75	0	29.75	0
		NOA	27.75	1.75	12.25	13.75
20	10	VOD	30	0	0	30
		BFD	50	0	50	0
		NOA	45	22.5	0	22.5
28	14	VOD	91	0	0	91
		BFD	126	0	126	0
		NOA 1	136.5	68.25	0	68.25
		NOA 2	136.5	63.375	9.75	63.375

Table 3.2: Three Dimensional Projection Property of Variance-Optimal Designs, Bias-Optimal Designs and Best NOAs in experiments with n=10, 12, 14, 18, 20, 28 where $m = n$=2

We use equation (1.1) to calculate all the values listed in Table 3.2. The orthogonal arrays of strength two are used as variance-optimal designs in experiments with 12, 20 and 28 runs, and hence their J_1- and J_2-components are zero. This is explainable as an OA of strength two is balanced and its columns are orthogonal to each other. Thus, only the J_3-component contributes towards the V_3. OAs of strength two are not available for run sizes 10, 14 and 18, where the corresponding variance-optimal designs are constructed using Lemma 1. In both situations, we can observe from Table 3.2 that the variance-optimal designs provide lower J_2-components. In contrast, J_3-components of bias-optimal designs are zero in every experiment, resulting in zero bias.

Nevertheless, the J_2-component of the best NOA in an experiment is lower than that of the bias-optimal design, indicating that it provides a variance lower than that of the bias-optimal design. Moreover, the best NOA provides a J_3-component lower than that of the variance-optimal design but considerably higher than that of the bias-optimal design. This contradictory relationship clearly explains the fact that the bias of a best NOA found in our study is lower than that of the corresponding variance-optimal design, while it is undoubtedly higher than the bias-optimal design. In conclusion, the above-explained bias-variance trade o among the set of three designs allows us to find three separate regions of C

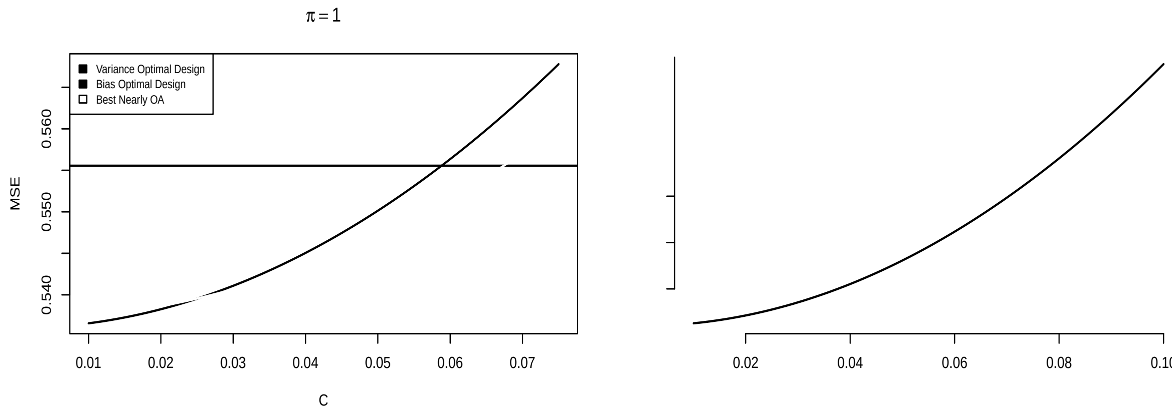


Figure 3.2: Comparison of the MSE for Designs With n=10 and m=5

According to Figure 3.2, the MSE of the variance-optimal design is the lowest for smaller values of C, which gradually increases to exceed the MSE of both the NOA and the bias-optimal design. Hence, the MSE of the variance-optimal design is the highest at higher values of C. The MSE of the best NOA slightly increases over the range of C and lies completely below the curve of the variance-optimal design after a certain point, while it eventually exceeds the constant MSE of the bias-optimal design. The MSE of both the variance-optimal design and the best NOA increases with the fraction of sparsity being considered in the study according to equation (2.1). For instance, when all the two-factor interactions are considered non-negligible, the two curves cross the curve of the best fold-over design at C values 0.059 and 0.067 respectively, whereas when only a half of the two-factor interactions are significant ($\pi = 1/2$), the intersections happen at slightly higher points, 0.083 and 0.095.

Figure 3.2 suggests three separate regions within each of which one out of the three designs performs better than the remaining two. Suppose the point where the curve of the variance-optimal design intersects that of the best NOA is indicated as C_1 and the intersecting point of curves of the best NOA and the bias-optimal design is named as C_2. Regardless of the fractions of sparsity being considered, the variance-optimal design and the best fold-over design perform the best when $C < C_1$ and $C > C_2$, respectively. The best NOA of the same order outperforms the two existing designs when $C_1 < C < C_2$. For example, the best NOA of order 10×5 outperforms the corresponding variance-optimal design and the bias-optimal design in the range of $0.025 < C < 0.068$ when none of the two-factor interactions are negligible, which means $\pi = 1$. Table 3.3 below summarizes the list of C_1 and C_2 values for all of the experiments listed in Table 3.2 for $\pi = 1; 1/2; 1/4; 1/8; 1/16; 1/32$. Figure 3.3 illustrates how the points C_1 and C_2 change when the fraction of sparsity increases. In conclusion, experimenters may use the listed regions to decide which design to use in their experiments.

- when $C < C_1$: Variance-optimal design is the best,
- when $C_1 < C < C_2$: Best NOA is the best,
- when $C > C_2$: Bias-optimal design (BFD) is the best.

n	m		$\pi = 1$	$\pi = 1/2$	$\pi = 1/4$	$\pi = 1/8$	$\pi = 1/16$	$\pi = 1/32$
10	5	C_1	0.025	0.036	0.051	0.071	0.101	0.143
		C_2	0.068	0.095	0.135	0.191	0.27	0.381
12	6	C_1	0.306	0.433	0.612	0.866	1.225	1.732
		C_2	0.339	0.479	0.677	0.957	1.354	1.915
14	7	C_1	0.064	0.091	0.128	0.182	0.257	0.363
		C_2	0.218	0.309	0.436	0.617	0.873	1.234
18	9	C_1	0.027	0.038	0.053	0.076	0.107	0.151
		C_2	0.111	0.158	0.223	0.315	0.446	0.63
20	10	C_1	0.048	0.068	0.095	0.135	0.19	0.269
		C_2	0.094	0.133	0.188	0.265	0.375	0.53
28	14	C_1	0.025	0.035	0.05	0.07	0.099	0.14
		C_2	0.045	0.064	0.09	0.128	0.181	0.255
		C_1	0.037	0.053	0.075	0.105	0.149	0.211
		C_2	0.043	0.061	0.087	0.122	0.173	0.245

Table 3.3: C_1 and C_2 at different values of π in experiments with $m = n/2$

Figure 3.3: C_1 and C_2 of the experiment of 10 5 as a function of π

3.2 Comparison of Variance-Optimal Designs and Best Nearly Orthogonal Arrays of Strength Three When Best Fold-Over Designs Do Not Exist

Best fold-over designs do not exist when the number of factors in an experiment is greater than half the number of runs being considered. Therefore, only variance-optimal designs are available to use in such situations. Our goal in this section is to introduce some alternative designs for practitioners to use over variance-optimal designs. We provide a list of the best NOAs of strength three for experiments in $10; 12; 14; 18; 20; 28$ runs with $m = n/2 + 1$ factors. We use the same procedure explained in Section 2.3 to construct best NOAs. For each experiment, we aim to compare the variance-optimal design with the best NOA in terms of the MSE to provide two separate regions of C, in which each of the designs performs better than the other design. Table 3.4 contains the V_3 values and the J-components of the two designs in each experiment.

n	m	Design	V_3	J_1 Component	J_2 Component	J_3 Component
10	6	VOD	10	1.875	1.5	6.625
		NOA	7.75	0	3.75	4
12	7	VOD	8.75	0	0	8.75
		NOA	9.25	0	6.25	3

According to Table 3.4, the J_2-component of the variance-optimal design is the lowest indicating the variance optimality. The best NOA contains the lowest J_3-component as it provides a smaller bias as compared to the variance-optimal design. To illustrate, we use the experiment of dimension 10 6 by changing $C = \tau = \sigma$ when $\pi = 1$.

n	m	C = –	VOD			NOA		
			Variance	Bias	MSE	Variance	Bias	MSE
10	6	0.025	0.65	0.008	0.657	0.675	0.007	0.682
		0.05	0.65	0.031	0.680	0.675	0.026	0.701
		0.1	0.65	0.122	0.771	0.675	0.103	0.778
		0.25	0.65	0.762	1.411	0.675	0.645	1.32
		0.5	0.65	3.046	3.696	0.675	2.58	3.255
		1	0.65	12.184	12.833	0.675	10.32	10.995
		2	0.65	48.735	49.384	0.675	41.28	41.955

Table 3.5: Comparison of the MSE for the Variance-Optimal Design and the Best NOA With 10 Runs and 6 Factors When All Two-Factor Interactions are Significant

According to Table 3.5, the variance of estimated main effects of both designs remains unchanged, being unaffected by τ. We can clearly observe that the MSE of the variance-optimal design is smaller at lower values of C and it gradually increases to exceed the MSE of the best NOA when C increases. That suggests that best NOAs are better than variance-optimal designs at higher values of C

Figure 3.4: Comparison of the MSE of Designs with $n = 10$ and $m = 6$

As can be seen in Figure 3.5, the C value increases as π decreases. That is because the bias decreases when π decreases according to the equation (2.1), and hence the MSE of the estimated main effects is also getting decreased. Thus, the C value required at the

Chapter 4

Concluding Remarks

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Appendix A

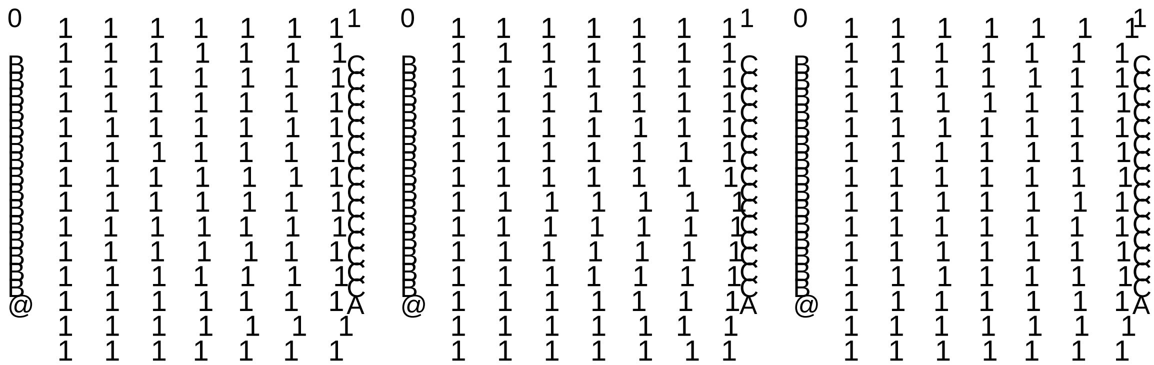
List of Designs

A.1 Designs used in the case where m = n=2; the Variance-Optimal Design, Bias-Optimal Design and the Best NOA respectively.

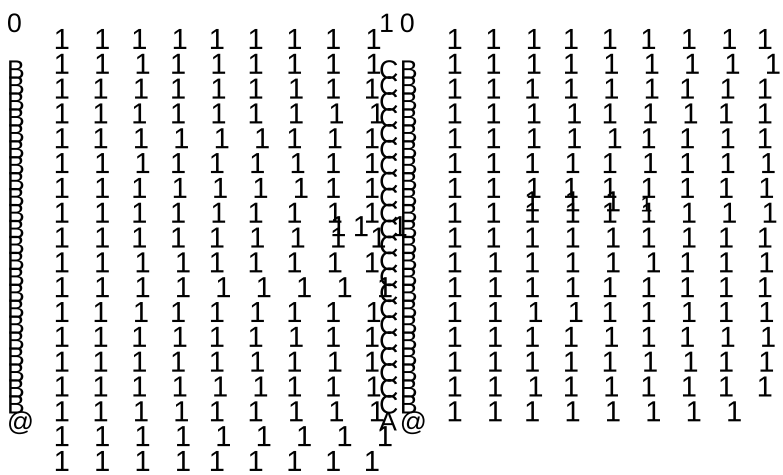
For n = 10 and

respectivelyandand0

For $n = 14$ and $m = 7$



For $n = 18$ and $m = 9$



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0                                              1
    1  1  1  1  1  1  1  1  1  1
B   1  1  1  1  1  1  1  1  1  1C
B   1  1  1  1  1  1  1  1  1  1C
B   1  1  1  1  1  1  1  1  1  1C
B   1  1  1  1  1  1  1  1  1  1C
B   1  1  1  1  1  1  1  1  1  1C
B   1  1  1  1  1  1  1  1  1  1C
B   1  1  1  1  1  1  1  1  1  1C
B   1  1  1  1  1  1  1  1  1  1C
B   1  1  1  1  1  1  1  1  1  1C
B   1  1  1  1  1  1  1  1  1  1C
B   1  1  1  1  1  1  1  1  1  1C
B   1  1  1  1  1  1  1  1  1  1C
B   1  1  1  1  1  1  1  1  1  1C
B   1  1  1  1  1  1  1  1  1  1C
B   1  1  1  1  1  1  1  1  1  1C
B   1  1  1  1  1  1  1  1  1  1C
@   1  1  1  1  1  1  1  1  1  1A
    1  1  1  1  1  1  1  1  1  1
    1  1  1  1  1  1  1  1  1  1
```

For n = 28 and m = 14

```
0
   1  1  1  1  1  1  1  1  1  1  1  1  1  1
B  1  1  1  1  1  1  1  1  1  1  1  1  1  1
B  1  1  1  1  1  1  1  1  1  1  1  1  1  1
B  1  1  1  1  1  1  1  1  1  1  1  1  1  1
B  1  1  1  1  1  1  1  1  1  1  1  1  1  1
B  1  1  1  1  1  1  1  1  1  1  1  1  1  1
B  1  1  1  1  1  1  1  1  1  1  1  1  1  1
B  1  1  1  1  1  1  1  1  1  1  1  1  1  1
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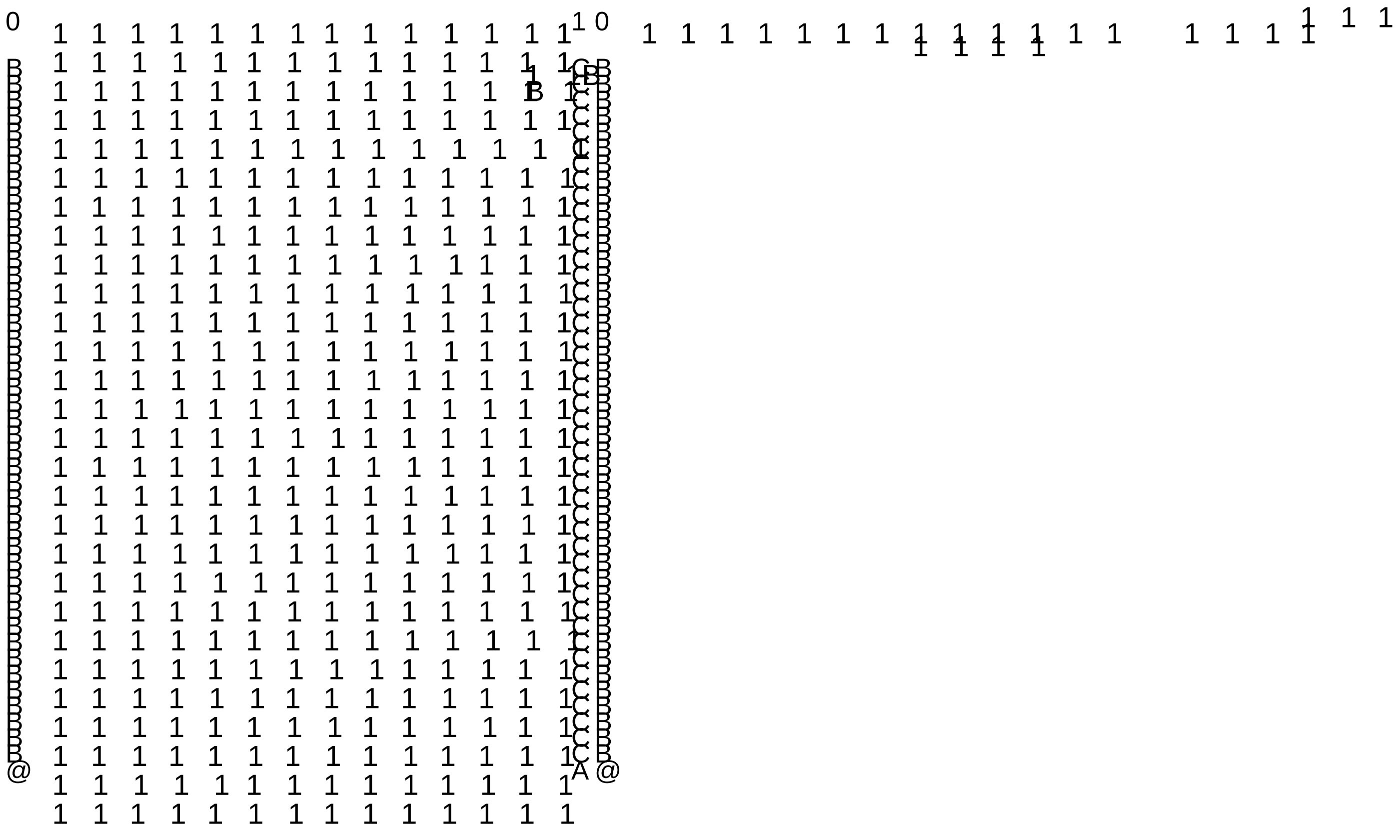
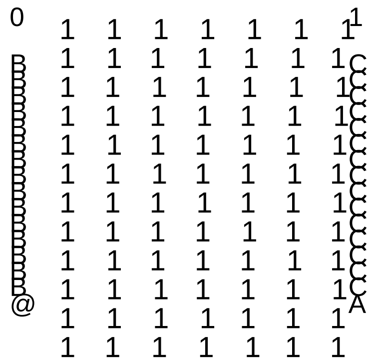
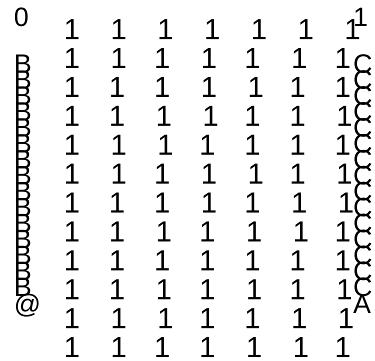
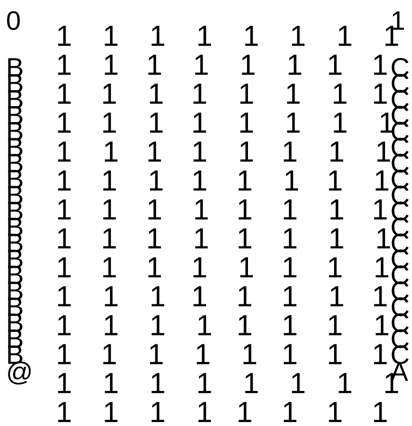


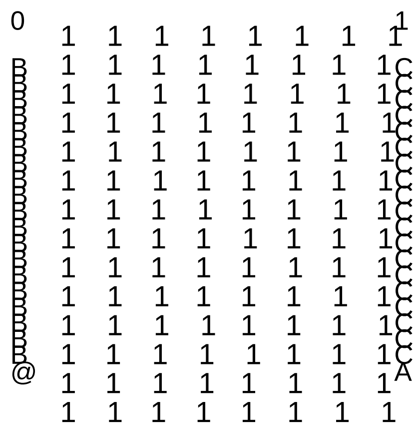
For n = 12 and m = 7



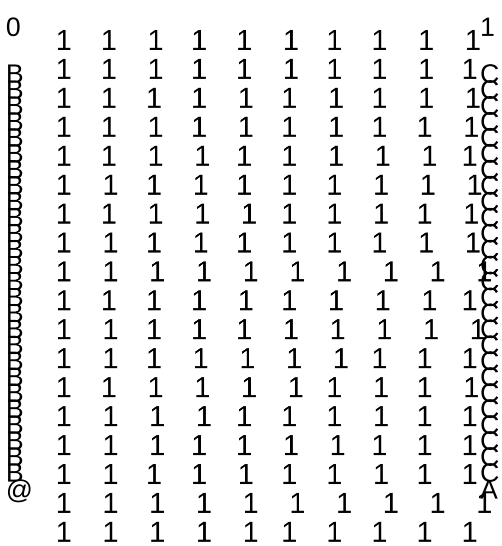


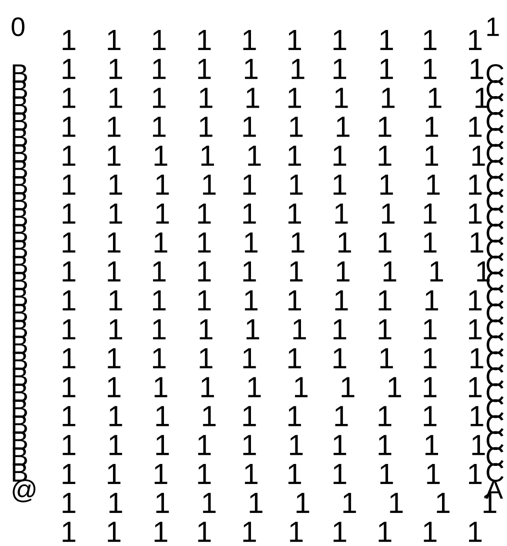
For n = 14 and m = 8



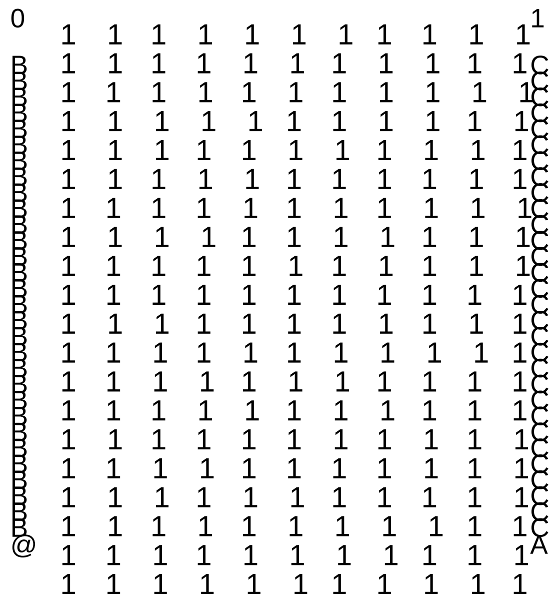


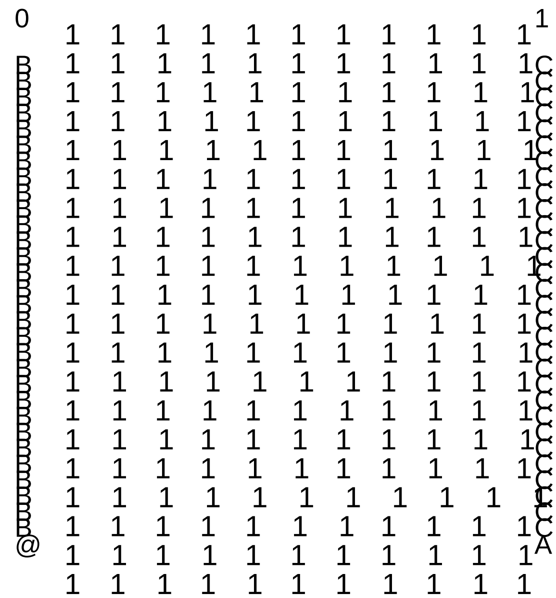
For n = 18 and m = 10





For n = 20 and m = 11





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