

Bayesian Sensitivity Analysis for Non-ignorable Missing Data in Longitudinal Studies

by

Tian Li

B.Sc. >University of Manchester>2014

Project Submitted in Partial Fulfillment of the
Requirements for the Degree of
Master of Science

in the
Department of Statistics and Actuarial Science
Faculty of Science

©Tian Li 2014
SIMON FRASER UNIVERSITY
Spring 2014

All rights reserved.

However, in accordance with the Copyright Act of Canada, this work may be reproduced without authorization under the conditions for Fair Dealing. Therefore, limited reproduction of this work for the purposes of private study, research, education, satire, parody, criticism, review and news reporting is likely to be in accordance with the law, particularly if cited appropriately.

Approval

Name: Tian Li

Degree:

Title:

Exam:

La
Se
As
Fa
&
As
Sta

Joan Hu
Sup
Pro

Ju
Int
Ass
Faculty of

Date Defended:

April 13 2011

Abstract

h?2 mb2 Q7 " v2bB M bi iBbiB+ H K2i?Q/b iQ ? M/H2 KBbbBM; / i BM
#2+QK2 TQTmH ` BM `2+2Mi v2 `bX AM i?Bb i?2bBb- r2 T`QTQb2 M
M HvbBb U"a V KQ/2H i? i ++QmMib 7Q` i?2 BM~m2M+2b Q7 KBbbB
2biBK iBQM Q7 i`2 iK2Mi 2z2+ib BM ` M/QKBx2/ +QMi`QH i`B Hb rBi
/ i X q2 BKTH2K2Mi i?2 K2i?Q/ mbBM; i?2 T`Q# #BHBbiB+ T`Q;` KKBM
TTHv Bi iQ / i 7`QK i?2 o M+Qmp2` i>QK2 Uo >V aim/v- r?B+? Bb ` `
i`B H i? i T`QpB/2/ ?QmbBM; iQ ?QK2H2bb T2QTH2 rBi? K2Mi H BHHM
Q7 "a iQ i?Qb2 7`QK M 2tBbiBM; " v2bB M HQM;Bim/BM H KQ/2H i?
BM i?2 Qmi+QK2X 6m`i?2`KQ`2- r2 /2KQMbi` i2 BM bBKmH iBQM bin
+QMb2`p iBp2 T`BQ` i? i /2b+`B#2b ` M;2 Q7 bbmK TiBQM b #Qmi i?
+`2/B#H2 BMi2`p Hb ? p2 ;`2 i2` H2M;i? M/ ?B;?2` +Qp2` ;2` i2 Q7 i?2
2tBbiBM; K2i?Q/b- M/ i? i b2MbBiBpBiv BM+`2 b2b b i?2 T2`+2Mi ;2
E2vrQ`/b", v2bB M K2i?Q/bc HQM;Bim/BM H M HvbBbc KBbbBM; / i c
aBKQM 6` b2` lMBp2`bBivc o M+Qmp2` i>QK2 bim/v

Dedication

h?Bb i?2bBb Bb /2H+K2/7i?2BMbHm/BM; UBM HT? #2iB+ H Q`/2`G
H2ii2` Q7 i?2 bm`M K2V

a `b? ` >Qbb2BMMB

a ; ` J2?i

> bb M EmHKB2

o bBHBb a2

>Qbb2BM a? `B}

ZBM;vm M 62M;

E2MM2i? *?mM;

GBHHB M u GX

JB+? 2H J+:Qp2`M

M;2H :` M /Qb

JB+? 2H C K2b _Q;2`b

M;mb GQ+F? `i

C vT` i T L B/m

r?Q ? p2 H m;?2/- r2Ti- r HF2/- `mM M/ TH v2/ 6A6 rBi? K2 Qp2` i?2
r?QK A ? p2 H2 `Mi 7 ` KQ`2 i? M A +QmH/ 2p2` ? p2 ?QT2/ 7Q`X

:` +B b- KX;Qb

Acknowledgements

A rQmH/HBF2 iQ 2tT`2bb Kv miKQbi;` iBim/2 iQ i?2 7QHHQrBM; BM/E

Kv bmt2`pS`Q7`2bbQ` G r`2M+2 J+* 7M`Hi2b+QM+2TiBQM Q7 i?Bb i?
M/ 7Q` ?Bb ;mB/ M+2- bmTTQ`i M/ BMbTB` iBQM i?`Qm;?Qmi Kv Ja
+mHKBM i2/ BM i?Bb }M H TB2+2 Q7 rQ`F- r?B+? rQmH/ MQi ? p2 #22M
BKTQ`i Mi bm;;2biBQMb M/ 2/Bib-

i?2 2t KBMBM;+QKKBiS`2Q72K#Q`bGQSMQ>72bbQ` CmHB M aQKK2`b
M/S`Q72bbQ` CBMFQ 7Q?iK2B` iBK2- T iB2M+2 M/ BMbB;?i7mH 722/

i?2 +`2 iQ`b. Q7 _Q#2`i :2MiH2M/S`Q72bbQ` _Qbb M7 FMv
Qi?2` +QMi`B#7nQQ`/2p2HQTBM;bm+? M K xBM;bi iBbiB+ H H M;m ;;

i?2 7QmM/2` Q7C_xiQx/BH B/ 2?_aim/BQ i2 KQ` K FBM; Bi bQ Km+?
2 bB2` iQ +Q/2 BM b B/ H M;m ;2-

M/ H bi #mi MQi H2 bi* Km M H B M; bq M/um2tm2-G7D` mM`2b2`p2/Hv
bmTTQ`iBM; 2p2`vi?BM; A /QX

Table of Contents

TT` Qp H

BB

List of Tables

h #H2 "XR .2b+`BTiBp2 bi iBbiB+b Q7 i?2 o M+Qmp2` i>QK2 / i b2i
H2bb BM/BpB/m Hb rBiX X2XiHXBXiXMX2XbX X X X X X X X X X

Chapter 1

Introduction

AM HQM;Bim/BM H bim/B2b- `2T2 i2/ Q#b2`p iBQMb Q7 i?2 b K2 b
T2`BQ/ Q7 iBK2X AM M B/2 H rQ`H/- `2b2 `+?2`b rQmH/ #2 #H2 iQ
i 2p2`v b+?2/mH2/ TQB MiX >Qr2p2`- Dmbi b KQbi i?BM;b BM HB72-
++Q`/BM; iQ TH M M/ KBbbBM; Q#b2`p iBQMb iQ p `BQmb 2ti2Mib /
rBb?2b Q7 2p2`vQM2 BMpQH p2/X

LmK2`Qmb K2i?Q/b ? p2 #22M /2pBb2/iQ /2 H rB8? i?2 Bbbm2 Q7 KBb

+7B`K`À

M iBQM H Q` +QKK2`+B H B Mi2`2bi- r?B+? Bb KQbiHv TTHB+ #H2 i
(kj)X q?2M / i `2 +QHH2+i2/ mbBM; bm`p2vb M/ B Mi2`pB2rb- MQM@`
Q7 +QM+2`M- r?2`2 T `iB+BT Mib 7 BH iQ `2bTQM/ iQ QM2 Q` KQ`2 B
HiQ;2i?2` U Hi?Qm;? bm+? + b2b + M #2 b 72Hv B;MQ`2/V 7Q` p `BQ
+2`i BM bm#D2+ib Q7 T`Bp i2 M im`2- bm+? b H2p2H Q7 BM+QK2-
MQM@`2bTQMb2 BM T `iB+mH `j;)`QmT Q7 T `iB+BT Mib (

JBbbBM;M2bb K v HbQ #2 BM /p2`i2MiHv + mb2/ #v`2b2 `+?2`b BM i
b +QMb2[m2M+2 Q7 KBbi F2b BM / i + QHH2`p2Qim2M2b/bi- i2Bb`v(T2
Q7 KBbbBM; / i + M #2`2H iBp2Hv 2 bv iQ `2+iB7v B7 i?2 Q`B;BM H

+h24>QbiH ° €° 84>p`@2

PM i?2 Qi?2` ? M/- / i `2 MQM@KQMqiQM2 KBbbBM;- Q` BMi2`KB
p iBQMb QM T `iB+BT Mi `2 K /2 7i2` i?2v KBbb T`BpBQmb / i +
Ai Bb TQbbB#H2 i? i i?2v #2+QK2 KBbbBM; ; BM- 2Bi?2` i2KTQ` `BH
T `iB+BT Mib `2 MQi i2+?MB+ HHv +QMbB/2`2/ /`QTQmib bBM+2 i?2v
7Q`2p2` i i?2 }`bi BMbi M+2 Q7 #b2M+2X

AM `2 HBiv- bi`B+iHv KQMqiQM2 KBbbBM; T ii2 rM-BbB M+2 M+QKK
Bi Bb mMHBF2Hv i? i HH T `iB+BT Mib rQmH/ #2 #H2 iQ /?2`2 iQ i?
KBbbBM; Mv BMi2`KBii2Mi TQBMib /m2 iQ T2`bQM H Q` Qi?2` `2 bQM

1.2.2 Classification of Missing Data

AM H M/K `F T T2` QM KBbbBM#B/Mi+(H bbB}2b i?2K BMiQ i?`22 + i
KBbbBM; +QKTH2i2Hv i` M/QK UJ* _V- KBbbBM; i` M/QK UJ _V M/
+ i2;Q`v r b H i2` M K2/ b KBbbBM; MQik}X M/QK UJL _V (

. i `2 J* _ B7 i?2 T`Q# #BHBiv Q7 7 BHBM; iQ Q#b2`p2 p Hm2 Bb
Q#b2`p2/ Q` mMQ#b2`p2/ p Hm2b Q7 i?2 `2bTQMb2 p `Bk}M2- Q` Mv C
?vTQi?2iB+ H 2t KTH2 +QmH/ #2 i? i- BM bim/v QM i?2 2z2+i Q7 /B
T `iB+BT Mib `QHH /B+2 iQ /2+B/2 r?2i?2` iQ ii2M/ K2 bm`2K2Mi
J* _ bbmKtiBQM- i?2 Q#b2`p2// i + M #2 +QMbB/2`2/ b ` M/QK b KT
/ i M/ i?2`2 Bb MQ #B b BM i?2 T ` K2i2` 2biBK i2bX

. i `2 b B/ iQ #2 J _ B7 i?2 T`Q# #BHBiv Q7 7 BHBM; iQ Q#b2`p2 p
Q7 Mv mMQ#b2`p2/ p Hm2b Q7 i?2 `2bTQMb2 p `B #H2- #mi/2T2M/2M
`2bTQMb2 p `B #H2 Q` bQK}Qi AM ip? BB #2H2V-(Bi Bb T2`? Tb KQ`2 BMi
BMi2`T`2i i?Bb ivT2 Q7 / i b óKBbbBM; +QM/BiBQM HHv i` M/QKôX
J _ rQmH/ Q++m` B7 T `iB+BT Mib rBi? HQR2` K2 bm`2/ +?QH2bi2`Q
KQ`2 HBF2Hv iQ KBbb bm#b2[m2Mi b2bbBQMb- Q` B7 T `iB+BT Mib QM
BM+HBM2/ iQ #2 #b2MiX

"v /2}MBiBQM- B7 / i `2 J* _- i?2v9)X2 "G b?QJf _ (M/ J _ `2
B;MQ` #H2 rBi?BM i?2 HBF2HB?QQ/ M/ " v2bB M7` K2rQ`Fb- r?2`2 b
B;MQ` #BHBiv Bb QMHv T}k}+ #H2 iQ J* _ (

q?2M i?2 T`Q# #BHBiv Q7 7 BHBM; iQ Q#b2`p2 p Hm2 Bb /2T2M/2
Bib2H7- Bi Bb FMQrM b JL _- Q` B}7Q`6KBiBm22KQ`b5BTMB}BT Mib iQ i
mT 7Q` b2bbBQM BM i?2 +?QH2bi2`QH bim/v b?QmH/ #2 ii`B#mi2/ iC
KBbbBM;M2bb Bb `2H i2/ iQ i?2B` +?QH2bi2`QH H2p2H BM i? i p2`v b2

#2+ mb2 i?2`2 Bb MQ BM7Q`K iBQM QM i?2 BM~m2M+2 Q7 i?2 KBbbBM
K2+? MBbK iQ #2 KQ/2HH2/X h`2 iK2Mi Q7 JL _ / i Bb i?2 7Q+mb Q7 i

Ai Bb p2`v /B{+mHi iQ b+2`i BM i?2 KBbbBM; / i K2+? MBbK BM
i?Qm;? b2p2` H K2i?Q/b ? p2 #22M /2p2HQT2/iQ i2bi 7Q`J* _ bbmKT
GBbiBM; M/ a+k)KBi iM/M.B; ;#2 (KQM; Qi?2`bX)1/b2`rb(b2b irQ T`Q@
+2/m`2b i? i /BbiBM;mBb? #2ir22M J* _ M/ J _- r?B+? `2Hv QM i?2 b
BM/2T2M/2M+2 Q7 i?2 KBbbBM; BM/B+ iQ` M/ i?2 Q#b2`p2/ +Qp`B i
;BbiB+ `2;`2bbBQMX LQirBi?bi M/BM;- BM ;2M2` H i?2`2 Bb MQ r v iQ
Q` JL _ 2tBbib 7Q` i?2v`2Hv QM BM7Q`K iBQM i? i Bb KBbbBM;- mM
Q#i BM2/ 7`QK MQM@`2bTQM/2M`b 7Q` p2`B}+ iBQM (

1.3 Review of Existing Methods in the Literature for Sensitivity Analysis of MNAR

Item t1 t# d

Chapter 2

Data Example

hQ KQiBp i2 Qm` /Bb+mbbBQM Q7 KBbbBM; / i BM HQM;Bim/BM H
/ i b2i /2b+`B#2/ #v S ii2`kQXmH2i2 HX +QM+2`M kNd ?QK2H2bb T2QT
o M+Qmp2`- "`BiBb? *QHmK#B r?Q T `iB+BT i2/ BM i?2 o M+Qmp2`
#2ir22M kyyN M/ kyRjX h?2 o > aim/v r b ` M/QKBx2/ +QMi`QH i`B H
T `iB+BT Mib rBi? K2Mi H BHHM2bb r2`2` M/QKHv HHQ+ i2/ iQ `2+2
Ui`2 iK2MiV Q` MQ ?QmbBM; U+QMi`QHV- M/ i?2M 7QHHQr2/ T`QbT2.
#Qmi ?2 Hi? Qmi+QK2b M/ b2`pB+2 mb2 rBi? `2T2jR2/ K2 bm`2K2Mib

2.1 Background: Homelessness in Canada and the At Home / Chez Soi Study

S2QTH2 BM i?2 h l ;`QmT`2+2Bp2/ MQ 7m`i?2` ?QmbBM; Q` bmTTQ`i
T `i7`QK i?2 2tBbiBM; b2`pB+2b 7Q` ?QK2H2bb BM/BpB/m Hb rBi? K2

h?2 T `iB+BT Mib r2`2 ` M/QKHv bbB;M2/ iQ i?2 i?`22 i`2 iK2Mi ;`C
HQR2/ T`QbT2+iBp2Hv 7Q` mT iQ irQ v2 `bX a22 6B;m`2 "XR 7Q` /B
bbB;MK2MiX . i r2`2 +QHH2+i2/ #v BMi2`pB2r- M/ 2 +? T `iB+BT M
iQ 8 iBK2b, QM+2 i # b2HBM2/ T`BQ` iQ ` M/QKBx iBQM- i?2M i?2M r
e@KQMi? BMi2`p HbX

a2p2` H bim/v ?vTQi?2b2b r2`2 7Q`KmH i2/- KQM; r?B+? r b i? i >C
? p2 TQbBiBp2 BM~m2M+2 QM i?2 [m HBiv Q7 HB72 UZQGV Q7 i?2 ?QK
BHBM2bb b +QKT `2/ iQ h IX h?mb- i?2 /2T2M/2Mi p `B #H2 BM Qm` M
Ub22 #2HQR 7Q` /2i BHbVX

h?2 T`2pBQmb M HvbBb #vk\$ +iQ Mb+QM22/ HX i(i?2 ZQG Q7 T `iB+BT
>6 BKT`Qp2/ bB;MB}+ MiHv KQ`2 i? M i? i Q7 T `iB+BT Mib BM h l i #C
TQbi # b2HBM2X aiBHH- M BKTQ`i MiTQi2MiB H HBKBi iBQM r b i? i

id	mMB[m2 M/ /2@B/2MiB}2/ B/ 7Q` T `iB+BT Mi
visit.number	h?2 pBbBi MmK#2`
visit.type	/2b+`BTiBQM Q7 i?2 pBbBi ivT2 U# b2HBM2- e KQM
visit.date	JQMi? M/ v2 ` Q7 pBbBi
csi	*QHq` /Q avKTiQK AM/2t Ub22 #2HqRv
qol	Zm HBiv Q7 HB72 b+Q`2 Ub22 #2HqRv
male	

;`QmTb M/ >6 + MMQi z2+ii?2 Qmi+QK2 i# b2HBM2X >Qr2p2`- bB;M
7QmM/ 7Q` e KQMi?b UT 4 yXyyk3V- Rk KQMi?b UT 4 yXyRReV M/ bH
KQMi?b UT 4 yXy3y3V- BM/B+ iBM; ?B;?2` K2 M ZQG b+Q`2b BM i?2 >6

6B;m`2 "XR b?Qrbi?2 K2 M Q7 i?2 ZQG i` D2+iQ`B2b BM >6 M/h IX A
M BM+`2 bBM;i`2M/ BM #Qi? ;`QmTb M/i?2 >6 ;`QmTb22b 7 bi2` BM
M BMi2`2biBM; +m`BQbBiv Bbi? ii?2 /Bz2`2M+2 #2ir22M i?2 +m`p2b i
h?mb- Hi?Qm;? >6 ? b TQbBiBp2 BKT +i QM ZQG +QKT `2/ iQ h l- i?2
i`2 iK2Mi z2+i TT2 `biQ #2 ;`2 i2bi2 `HB2` BM i?2 7QHHQr@mT T2`
K2 M ZQG b+Q`2 BM h l r b HbQ Q#b2`p2k ~~SM SHIBZ~~ 2HOM2THHM (BQM

* ? T i 2 ` j

G Q M ; B i m / B M H M H v b B b i ?
A ; M Q ` 2 b J B b b B M ; . i B M i ? 2
P m i + Q K 2 o ` B # H 2 Z m H B i v Q 7

_ 2 + H H i ? i i ? 2 / i + Q M b B b i b Q 7 H Q M ; B i m / B M H K 2 b m ` 2 K 2 M i b Q 7
n = 297 T ` i B + B T M i b - r ? Q r 2 ` 2 7 Q H H Q r 2 / T ` Q b T 2 + i B p 2 H v 7 Q ` k 9 K Q M i
r b ` M / Q K H v H H Q + i 2 / i Q 2 B i ? 2 ` i ` 2 i K 2 M i U > Q m b B M ; 6 B ` b i U > 6 V V Q
l b m H U h I V V X h ? 2 v r 2 ` 2 i ? 2 M B M i 2 ` p B 2 r 2 / m T i Q 8 i B K 2 b U # b 2 H B M 2
R 3 K Q M i ? b M / k 9 K Q M i ? b V - M / / 2 i B H 2 / / i r 2 ` 2 ` 2 + Q ` / 2 / X h ? 2 / 2 T 2
i ? 2 M H v b B b r b ` 2 T 2 i 2 / K 2 b m ` 2 b Q 7 Z m H B i v Q 7 G B 7 2 U Z Q G V X

S ` 2 p B Q m b H v - H Q M ; B i m / B M H / i M H v b 2 b Q 7 i ? 2 o > / i r 2 ` 2 + Q M
S i i 2 ` b Q M 2 i 9 X H X ? (2 m i ? Q ` b m b 2 / H B M 2 ` K B t 2 / 2 z 2 + i b ` 2 ; ` 2 b b B Q M i C
+ B i B Q M # 2 i r 2 2 M i ? 2 / B z 2 ` 2 M i i v T 2 b Q 7 > 6 M / i ? 2 M Q ` K H H v / B b i ` B # m
` 2 ; ` 2 b b B Q M M H v b B b - i ? 2 m i ? Q ` b B M + H m / 2 / i B K 2 U / B b + ` 2 i 2 e K Q M i
i 2 ` K b # 2 i r 2 2 M i B K 2 M / b i m / v ` K - r ? B + ? + T i m ` 2 i ? 2 i ` 2 i K 2 M i 2 z 2 + i b
i ? 2 K m H i B p ` B # H 2 K Q / 2 H i ? 2 m i ? Q ` b / D m b i 2 / 7 Q ` # b 2 H B M 2 + Q p ` B
M / Q i ? 2 ` p ` B # H 2 b b m + ? b ? Q m b B M ; b i i m b i # b 2 H B M 2 M / / m ` i B C
M 2 b b X S i i 2 ` b Q M 2 Z i Q H H / (i ? i > 6 r b b b Q + B i 2 / r B i ? b B ; M B } + M i H v ; `
b + Q ` 2 b b + Q K T ` 2 / i Q h I X

h ? 2 b i i B b i B + H B b b m 2 i ? i K Q i B p i 2 b i ? B M i 2 ` b 2 K B i 2 M i B Q D 2 # i B b i ? 2
Q 7 Z Q G K 2 b m ` 2 K 2 M i b X b / 2 b + B T # 2 / B B H T ? M T b 2 ` b b B ; M 2 / i Q h l r 2 ` 2 H 2
H B F 2 H v i Q # 2 B M i 2 ` p B 2 r 2 / B M i ? 2 7 Q H H Q r @ m T T 2 ` B Q / # 2 + m b 2 Q 7 H
B M i ? 2 b i m / v X u 2 i b m + ? K B b b B M ; M 2 b b / Q 2 b M Q i + Q M b i B i m i 2 H Q b b i Q 7
b F B T T 2 / Q M 2 Q ` K Q ` 2 B M i 2 ` p B 2 r b # m i r 2 ` 2 B M i 2 ` p B 2 r 2 / ; B M H i 2 ` B M

?2`2 Bb i?2M ?Qr i?Bb KB;?i ? p2 z2+i2/ i?2 `2bmHibX 6Q` 2t KTH2-
 rBi? rQ`b2 ?2 Hi? r2`2 KQ`2 HBF2Hv iQ #2 HQbi- i?2 ZQG i` D2+iQ`v E
 #B b2/ b `2bmHi Q7 ii`BiBQM Q7 i?2 bB+F2bi T iB2MibX

AM i?Bb b2+iBQM Q7 i?2 i?2bBb- r2 #2;BM #v `2THB+ iBM; i?2 HBM
 Q7 S ii2`bQMk2imHBM; " v2bB M K2i?Q/b BKTH2K2Mi2/ BM i?2 bQ7ir`
 M HvbBb `2bmHib rBHH b2`p2 b TQBMi Q7 +QKT `BbQM rBi? i?2 bm
 KQ/2H i?2 KBbbBM; ZQG b+Q`2 /B`2+iHv mbBM; MQM@B;MQ` #H2 KE

3.1 Model

"mBH/BM; mTQM i?2 M HvbBbkQ7 S iT22bQM2i" MxbB M HBM2 `KBt
 2z2+i KQ/2Hb iQ ++QmMi 7Q` +Q`2H iBQM BM `2T2 i2/ K2 bm`2b Q
 aim/vX 6Q` 2 b2 Q7 `272`2M+2- i?Bb KQ/QM {BHQ/ #H M2K2mb2 i?2
 M [p2Hv B;MQ`2b i?2 `QH2 Q7 KBbbBM; / i BM i?2 M HvbBbX

3.1.1 Variables and Notation

$G_{2ij} \#2 i?2 [m HBiv Q7 HB^7 T biBQ`BT7 Mi BM2i+Q`/- r?2`2$
 1 2 297 M/ = 1 2 3 4 5 `2T`2b2Mib # b2HBM2 M/ }`bi iQ 7Qm`i? pBbBi`2

$G_{2i} \#2 M BM/B+ iQ` p `B #H2 7Q` i?2 ;`Q m T i B+BT+ MB-Qm Q7 ii? 2$

$$\begin{aligned}
 & i = 1 B^h T `iB+BT Mi Bb BM i?2 >6 ;`Q m T \\
 & = 0 B^h T `iB+BT Mi Bb BM i?2 h l ;`Q m T
 \end{aligned}$$

LQi2 i? i BM i?2 o > bim/v i?2 i`2 iK2Mi HHQ+ iBQM r b }t2/ Qp2` iB
 i?2 h l T `iB+BT Mib r2`2 MQi T`2p2Mi2/ 7`QK }M/ ?QmbBM; QM i?2B` C
 HBKBi iBQM U/Bb+m v B/bBiM i?2 i`2 i BM/BpB/m Hb KB;?i BM 7 +i ? p2
 ?QmbBM; i?`Qm;? Qi?2` K2 MbX

r?2`2()+ Bb MQ`K H /Bbi`B#miBQM/rpB`BKM?Mi Bb i`mM+ i2/ iQ
#2 bi`B+iHv TQbN0B02X`BQ` Bb bbB;M2/ iQ i?2 ` M/QK 2z2+i- iBK2 2
i`2 iK2Mi #v T2`BQ/ BMi2` +iBQM iQ`2~2+i i?2`2 HBbiB+ p`B iBQM B
(ky- R9y) BM i?2 / i b2iVX >Qr2p7(0-10000) rBb2mTb`B/Q7Q` i?2 p`B M+2b
2 M/2 /m2 iQ H +F Q7 T`BQ` BM7Q`K iBQM- BM/B+ iBM; i? i H`;2
TH mbB#H2X

3.3 Computations Using Stan

ai M Bb T`Q# #BHBbiB+ T`Q;` KKBM; H M;m ;2 7Qj)X q2B@M bi iBb
i2M BM *YY- Bi Bb mb2/ iQ bT2+B7v bi iBbiB+ H KQ/2Hb M/ BKTH2K2
* `HQ UJ*J*V K2i?Q/b- ;` /B2Mi@# b2/ p`B iBQM H" v2bB M K2i?Q/b
QTiBKBx iBQM 7Q` T2M HBx2/ K tBKmK HBF2HB?QQ/ 2biBK iBQMX
100000)

++Q`/BM;Hv- r2 M Hvx2/i?2 o > / i b2i M/` M i?2 > KBHiQMB M J

2z²+i 2biBK i2b' ~~EM~~ h #H2

Chapter 4

Bayesian Sensitivity Analysis (BSA) for Non-ignorable Missing Data

4.1 Model

Air b b?QrM BM i?2 T`2pBQmb +? Ti2` i? i bQK2 Q7 i?2 ZQG b+Q`
M im` H [m2biBQM Q7 Bmi2`2bi Bb r?2i?2` i?2 BM+HmbBQM Q7 i?2b2
i?2 +QM+HmbBQM b /` rM 7`QK i?2 M Hvb2b mbBM; QMHv 2tBbiBM; /
Q7 BM~m2M+2X h?2 F2v Bbbm2 Bb r?2i?2` i?2 / i `2 KBbbBM; i` M
KBbbBM;X >Qr2p2`- i?Bb + MMQi #2 b+2`i BM2/ 7`QK Q#b2`p2// i #
?Qr i?2 KBbbBM; / i /Bz2`b 7`QK i?2 Q#b2`p2// i X

hQ #2ii2` mM/2`bi M/i?2 BKT +i Q7 MQM@B;MQ` #H2 KBbbBM; / i - r
7`QK b2MbBiBpBiv M He)KBp2TT2`Q TQ b 2B p2Q p2H K2i?Q/QHQ;v + HH2/
b2MbBiBpBiv M HvbBbô U"a V iQ 2tTHQ`2 b2MbBiBpBiv iQ MQMB;MQ
Bb iQ T`QTQb2 KQ/2H 7Q` i?2 +QKTH2i2 / i UQ#b2`p2/ M/ mMQ#b2`
MQM@B/2 M2M) b B H2p Biv T ` K2i4Ð Ô" DÐ Bi•Ÿw) p à Bi R

+ MMQi /BbiBM;mBb? #2ir22M /Bz2`2Mi KQ/2Hb 2p2M bvK TiQiB+ HHv
 mMmbm H bi iBbiB+ H T`QT2`iB2bX

q2 #2;BM i?Bb +? Ti2` rBi? T`QTQbBM; KQ/B}2/ T ii2`M KBtim`2 K
 7Q` i?2 2z2+i Q7 KBbbBM;M2bb QM i?2 Q#b2`pe)XQMb- 7QHHQrBM; >Q

4.1.1 Missing Data Model

G2_{ij} #2 i?2 [m HBiv Q7 Hth T2`b B QBZ 7Qth BM +Q` /- r?2`2 B 4 R- k- j-
 XXX- kNd M/ D 4 R- k- j- 9- 8`2T`2b2Mib # b2HBM2 M/ }`bi iQ 7Qm`i?
 B/2 H b+2M`BQ r?2`2 i?2`2 Bb MQ KBbbBM;8/4i R-932`Q#b2`p/ i B Q M N X

G2_i #2 M BM/B+ iQ` p`B #H2 7Q` i?2 ;`th m TiB+BT+MB Qm Q? ii? 2

$$\begin{aligned}
 & i = 1 \ B \# \ T \ `iB+BT \ Mi \ Bb \ BM \ i?2 \ >QmbBM; \ 6B \ `bi \ ; \ QmT \\
 & = 0 \ B \# \ T \ `iB+BT \ Mi \ Bb \ BM \ i?2 \ h`2 \ iK2Mi \ b \ l \ m \ H \ ; \ QmT
 \end{aligned}$$

6Q` 2 +? T`iB+BTthMiB Q i/?-2r2 +`2 i2thV_j Q7 H Q`M;i? 9 iQ`2T`2b2Mi i?2
 MmK#2` Q7 pBbBi- bm+? i? i

$$\begin{aligned}
 \widetilde{V}_j &= [0 \ 0 \ 0 \ 0] \ 7Q` \ # \ b \ 2 \ H \ B \ M \ \# \ U \\
 &= [1 \ 0 \ 0 \ 0] \ 7Q` \ i?2 \ } \ `bi \ p \ Bb \ Bi- \\
 &= [0 \ 1 \ 0 \ 0] \ 7Q` \ i?2 \ b \ 2 \ + \ QM \ / \ p \ Bb \ Bi- \\
 &= [0 \ 0 \ 1 \ 0] \ 7Q` \ i?2 \ i?B \ ` \ / \ p \ Bb \ Bi- \\
 &= [0 \ 0 \ 0 \ 1] \ 7Q` \ i?2 \ 7Qm`i? \ p \ Bb \ Bi
 \end{aligned}$$

G2_{ij} #2 M BM/B+ iQ` p`B #H2 7Q` KBbbB_{ij}M;M2+b2iBM i?2 Qmi+QK2

$$\begin{aligned}
 I_{ij} &= 1 \ B \# \ i \ j \ Bb \ KBbbBM; - \\
 &= 0 \ B \# \ i \ j \ Bb \ Q \ # \ b \ 2 \ ` \ p \ 2 \ /
 \end{aligned}$$

LQ_i2 i? ii?2 [m I_{ij} MiBiBth V_j `2 Hr vb Q#b2`p2/ 7Q` HHTQbbB#H2 +QK
 Q7 B M/ DX

4.1.2 Pattern Mixture Model for Missing Data

$$I_{ij} | i \tilde{V}_j = (I_{ij} | i \tilde{V}_j) (I_{ij} | i \tilde{V}_j) \quad U9XR V$$

$$I_{ij} | i \tilde{V}_j \sim N(\theta_i + \tilde{V}_j^T \tilde{\beta} + (\tilde{V}_j^T \tilde{\beta})^2) \quad U9Xk V$$

$$\tilde{\beta} = [\beta_{vm} \beta_{vm} \beta_{vm} \beta_{vm}]$$

* Q M b 2 M H M b ? t - ñ ' ø V M ñ 1 ñ # ' ø l ò • ÿ ' - Z Q G b + Q ` 2 b 7 Q `

#2HB2p2 i? ii?2 BKT +i Q7 ii`BiBQM QMi?2 bim/v +? M;2/ Qp2` iBK2X

$G_{2i_1} = (I_{ij} = 1 | = 1) \# 2 i^2 T^Q \# \# BHBiv Q7 KBbbBM; M2bb BM i^2 > 6$
 $0 = (I_{ij} = 1 | = 0) \# 2 i^2 T^Q \# \# BHBiv Q7 KBbbBM; M2bb BM i^2 h l ; ^Qr$
 $2tT2+i2/ ZQG b+Q^2 7Q^i^2 h l ; ^QmT Bb$

$$\begin{aligned}
 \mathbf{I}[| = 0 \widetilde{V}] &= \mathbf{I}[| = 0 \widetilde{V}^{I=1}]_0 + \mathbf{I}[| = 0 \widetilde{V}^{I=0}](1 - 0) \\
 &= (0 + \widetilde{V}^T(\widetilde{\beta} + \widetilde{\beta}))_0 + (0 + \widetilde{V}^T(\widetilde{\beta}))_0(1 - 0) \quad U9X8V \\
 &= 0 + \widetilde{V}^T(\widetilde{\beta} + 0\widetilde{\beta})
 \end{aligned}$$

$r^2 \cdot 2\widetilde{\beta} \quad Bb i^2 \# B b b^? B7iX$

$PM i^2 Qi^2 \cdot ? M/- i^2 2tT2+i2/ ZQG b+Q^2 7Q^i^2 > QmbBM; 6B^bi ;$

$$\begin{aligned}
 \mathbf{I}[| = 1 \widetilde{V}] &= \mathbf{I}[| = 1 \widetilde{V}^{I=1}]_1 + \mathbf{I}[| = 1 \widetilde{V}^{I=0}](1 - 1) \\
 &= (0 + \widetilde{V}^T(\widetilde{\beta} + \widetilde{\beta} + \widetilde{\beta}))_1 + (0 + \widetilde{V}^T(\widetilde{\beta} + \widetilde{\beta}))_0(1 - 1) \quad U9XeV \\
 &= 0 + \widetilde{V}^T(\widetilde{\beta} + \widetilde{\beta} + 1\widetilde{\beta})
 \end{aligned}$$

$r^2 \cdot 2\widetilde{\beta} \quad Bb i^2 \# B b b^? B7iX$

$h^2 \cdot 27Q^2 i^2 Qp2^ \quad HH i^2 iK2Mi 2z2+i + M \# 2 b^? QrM iQ \# 2$

$$\mathbf{I}[| = 1 \widetilde{V}] - \mathbf{I}[| = 0 \widetilde{V}] = \widetilde{V}^T(\widetilde{\beta} + (1 - 0)\widetilde{\beta}) \quad U9XdV$$

$h^? Bb ; Bp2b \quad bBKTH2 \quad M HviB+ H 7Q^KmH 7Q^2tTHQ^BM; b2MbBiB$
 $/ i X \quad q2 b22 \widetilde{\beta}^? iBb i^2 Q\#b2^p2/ i^2 iK2Mi 2z2+i \quad M/ //BiBQM HHv- i$
 $(1 - 0)\widetilde{\beta} \quad \widetilde{B}b i^2 id\mathcal{D}! \alpha Mr \quad -Mid^ \cdot \delta \quad id\mathcal{D}! \alpha F \cdot e\check{N}F 2bb BM i^2 20$

"mBH/BM; QMj*7QTi2?2 " v2bB M HBM2 ` 2z2+ib KQ/2H- r2 bbB;M i
T`BQ` /Bbi`B#miBQMb 7Q` i?2 T ` K2i2`b-

N (0;100)
2 N (0;100000)*
2 0 13
e_v N 0; 100 0 0 0
4 @ 0 100 0 0
0 0 100 0
0 0 0 100
2 0 13
e_vx N 0; 100 0 0 0
4 @ 0 100 0 0
0 0 100 0
0 0 0 100
2 N (0;100000)*:

U9X3V

6Q`_vm - r2 mb2/ irQ /Bz2`2Mi TT`Q +?2b iQ bbB;M T`BQ` /Bbi`B#
bbB;M2/ bT2+B}+ }t2/_m pUH Q 2 B_mj B b MQi ` M/QK p `B #H2VX a2p2M p
` M;BM; 7`QK @jy iQ jy r2`2 b2H2+i2/ iQ +Qp2` #`Q /_vm M;2 Q7 TQb
h?2`2 7i2`- r2 `2T2 i2/ i?2 M HvbBb M/ bMB;7(Q236 30) XBQ?mb
i?2}`bi M HvbBb Bb +H bbB+ ób2MbBiBpBiv M HvbBb_ m BM i?2 b2Mb
Bb }t2/ i bT2+B}+ p Hm2 i? i ` M;2b Qp2` ;`B/ BM Q`/2` iQ bim/v i?2
Q7 i?2 `2bmHib- r?2`2 b i?2 b2+QM/ M HvbBb Bb +H bbB+ " v2bB M
r2 KQ/2H Qm` T`BQ` #_m Bb27m M#BQr i K /Bbi`B#miBQM X

aBM+2 i?2 ZQG b+Q`2b `2 BM i?2 ` M;2 (ky- R9y)- Bi rQmH/ #2 `2
N(0;100) T`BQ` iQ i?2 ` M/QK 2z2+i- iBK2 2z2+i M/ i`2 iK2Mi #v T2`B
> Qf4y850Tdq(#100)TJ/F6 10.9069>]TJ.3592>-435<002a005100435

4.3 Computations Using Stan

```

AM i?2 M HvbB bstrab BTM+ F?;22- i?2 HBbi Q7 / i r b }`bi /2}M2/-

int n;                UMmK#2` Q7 T `iB+BT MibV
int nobs;             UMmK#2` Q7 Q#b2`p2/ `2+Q`/bV
int nmis;             UMmK#2` Q7 KBbbBM; `2+Q`/bV
int id[nobs+nmis];   Up2+iQ` Q7 T `iB+BT Mi B/b Q7 H2M;i? 8MV
real yobs[nobs];     Up2+iQ` Q7 Q#b2`p2/ ZQG b+Q`2b Q7 H2M;i? M
real x[nobs+nmis];   Up2+iQ` Q7 ;`QmT HHQ+ iBQM b Q7 H2M;i? 8M
matrix[nobs+nmis, 4] v; UK i`Bt Q7 pBbBi MmK#2`bV
real betavm          U#B b /m2 iQ KBbbBM;M2bbV

7QHHQr2/ #v HBbi Q7 i?2 T ` K2i2`b-

real theta[n];       U` M/QK 2z2+ib BM HBM2 `KBt2/ 2z2+ib KQ
real mu_theta;       UK2 M Q7 i?2 ` M/QK 2z2+ibV
vector[4] betav;     UiBK2 2z2+iV
vector[4] betavx;    UiBK2 2z2+iV
real <lower=0> sigma_theta; Ub/ Q7 i?2 ` M/QK 2z2+ibV
real <lower=0> sigma;   Ub/ Q7 i?2 ZQGV
real ymis[nmis]      UKBbbBM; / i iQ #2 BKTmi2/V

M/ i?2 T`BQ`b bT2+B}2/ BM i?2 T`2pBQmb b2+iBQMX

h?2 KBbbBM; / i r b BKTmi2/ BM i?2 T`2pBQmb b2+iBQMX

for (i in 1:(nmis))
  ymis[i] ~ normal(v[i+nobs] * (betav + betavm) + (v[i+nobs] * betavx)
* x[i+nobs] + theta[id[i+nobs]]], sigma);

rBi? i?2 Q#b2`p2/ / i BM MQi?2` HQQT-

for (i in 1:(nobs))
  yobs[i] ~ normal(v[i] * betav + (v[i] * betavx) * x[i] + theta[id[i]],
sigma);

q2 ` M i?2 > KBHiQMB M JQMi2 * `HQ H;Q`Bi?K 7Q` kyyy Bi2` iBQ
RyyyX a KTH2 +QMp2`;2M+2 r b QM+2 ; BM bb2bb2/ mbBM;i?2 2z2+i
`2/m+iBQM 7 +iQ`- r?B+? `2 miQK iB+ HHv ;2M2` i2/ BM ai MX

```

4.4 Results

4.4.1 Sensitivity Analysis Where β_{vm} is Fixed over a Specific Grid of Values

$a_2 M b B i B p B i v \quad M \quad H v b B b \quad \beta_{vm} \quad Q \quad H \quad 2 \quad b \quad r \quad B i T \quad 2 \quad \# \quad B \quad \} + ; \quad B / \quad Q \quad 7 \quad p \quad H m \quad 2 \quad b \quad T \quad Q / m + 2$
 $K \quad i \quad 2 \quad b \quad Q \quad 7 \quad T \quad \quad K \quad 2 \quad i \quad 2 \quad \quad b \quad b \quad / \quad B b \quad T H \quad v \quad 2 \quad X \quad B \quad M \quad b \quad \beta_{vm} \quad \# \quad 2 \quad 2 \quad \quad b \quad 2 \quad b \quad 7 \quad \quad Q K \quad j y \quad U K \quad 2 \quad M \quad Z \quad Q \quad G$
 $7 \quad Q \quad \quad K B \quad b b \quad B M ; \quad T \quad \quad i B + B T \quad M i b \quad j y \quad T \quad Q \quad B \quad M i b \quad ? \quad B ; \quad ? \quad 2 \quad \quad i ? \quad M \quad Q \quad \# \quad b \quad 2 \quad \quad p \quad 2 / \quad T \quad \quad i B + B T$
 $7 \quad Q \quad \quad K B \quad b b \quad B M ; \quad T \quad \quad i B + B T \quad M i b \quad j y \quad T \quad Q \quad B \quad M i b \quad H \quad Q \quad r \quad 2 \quad \quad i ? \quad M \quad Q \quad \# \quad b \quad 2 \quad \quad p \quad 2 / \quad T \quad \quad i B + B T$
 $K \quad Q \quad M \quad Q \quad i \quad Q \quad M \quad B + \quad B \quad M + \quad \quad 2 \quad \quad b \quad 2 \quad \quad B \quad M \quad i \quad ? \quad 2 \quad 2 \quad b \quad i \quad B \quad K \quad i \quad 2 \quad b \quad Q \quad 7 \quad \quad H \quad H \quad \beta \quad Q \quad x \quad m \quad \quad i \quad \quad 2 \quad \quad i \quad K \quad 2 \quad M \quad i \quad 2 \quad z \quad 2 \quad + \quad i$

$I \quad T \quad Q \quad M \quad + \quad H \quad Q \quad b \quad 2 \quad \quad B \quad M \quad b \quad T \quad 2 \quad + \quad i \quad B \quad Q \quad M \quad r \quad 2 \quad \beta_{vm} \quad \quad B \quad b \quad 2 \quad 2 \quad b \quad B \quad i \quad B \quad p \quad 2 \quad M \quad Z \quad Q \quad G \quad 7 \quad Q \quad \quad K B \quad b b \quad B M ; \quad \quad i ? \quad M \quad Q \quad \# \quad b \quad 2 \quad \quad p \quad 2 / \quad V - \quad i \quad ? \quad 2 \quad \quad 2 \quad \quad B \quad b \quad M \quad Q \quad Q \quad \quad H \quad B \quad i \quad i \quad H \quad 2 \quad \quad b \quad B ; \quad M \quad B \quad \} + \quad M \quad i \quad i \quad \quad 2 \quad \quad i \quad K \quad 2 \quad M \quad i \quad 2 \quad z \quad 2 \quad + \quad i \quad \quad i ? \quad 2 \quad T \quad 2 \quad \quad + \quad 2 \quad M \quad i \quad ; \quad 2 \quad \quad Q \quad 7 \quad K B \quad b b \quad B M ; \quad M \quad 2 \quad b \quad b \quad B \quad b \quad ? \quad B ; \quad ? \quad 2 \quad \quad B \quad M \quad i \quad ? \quad 2 \quad \quad h \quad l \quad ; \quad \quad Q \quad m \quad T - \quad ? \quad B ; \quad ? \quad 2 \quad \quad T \quad \quad i B + B T \quad M i b \quad r \quad Q \quad m \quad H / \quad \quad 2 \quad b \quad m \quad H \quad i \quad B \quad M \quad \quad b \quad K \quad H \quad H \quad 2 \quad \quad + \quad i \quad m \quad H \quad / \quad B \quad z \quad 2 \quad \quad 2 \quad M + \quad 2 \quad \quad \# \quad 2 \quad i \quad r \quad 2 \quad 2 \quad M > \quad 6 \quad i ? \quad m \quad b \quad Q \quad p \quad 2 \quad \quad \quad B / \quad B \quad M ; \quad M \quad v \quad T \quad 2 \quad \quad + \quad 2 \quad B \quad p \quad 2 / \quad i \quad \quad 2 \quad \quad i \quad K \quad 2 \quad M \quad i \quad 2 \quad z \quad 2 \quad + \quad i \quad X$

$P \quad M \quad i \quad ? \quad 2 \quad + \quad Q \quad M \quad i \quad \quad \quad v - \quad \beta_{vm} \quad 2 \quad p \quad i \quad B \quad p \quad 2 \quad b \quad U \quad Z \quad Q \quad G \quad 7 \quad Q \quad \quad K B \quad b b \quad B M ; \quad H \quad Q \quad r \quad 2 \quad \quad i ? \quad M \quad Q \quad \# \quad b \quad 2 \quad \quad i ? \quad i \quad i \quad ? \quad 2 \quad + \quad i \quad m \quad H \quad / \quad B \quad z \quad 2 \quad \quad 2 \quad M + \quad 2 \quad \quad B \quad M \quad Z \quad Q \quad G \quad \# \quad 2 \quad i \quad r \quad 2 \quad 2 \quad M > \quad 6 \quad M / \quad h \quad l \quad B \quad b \quad H \quad \quad ; \quad 2 \quad \quad i ? \quad M \quad Q \quad K \quad T \quad H \quad B \quad 7 \quad v \quad B \quad M ; \quad i \quad ? \quad 2 \quad i \quad \quad 2 \quad \quad i \quad K \quad 2 \quad M \quad i \quad 2 \quad z \quad 2 \quad + \quad i \quad X \quad L \quad i \quad m \quad \quad H \quad H \quad v - \quad B \quad i \quad b \quad 2 \quad b \quad i \quad B \quad K \quad i \quad 2 \quad b \quad \quad 2 \quad \quad K \quad m + \quad ? \quad K$

$h \quad ? \quad 2 \quad + \quad b \quad 2 \quad \quad Q \quad 7 \quad y \quad B \quad b \quad i \quad \quad B \quad p \quad B \quad H - \quad b \quad r \quad 2 \quad r \quad 2 \quad \quad 2 \quad \quad b \quad B \quad K \quad T \quad H \quad v \quad \quad b \quad b \quad m \quad K \quad B \quad M ; \quad i ? \quad i \quad i \quad ? \quad 2 \quad \quad 2 \quad \quad B \quad b \quad Z \quad Q \quad G \quad \# \quad 2 \quad i \quad r \quad 2 \quad 2 \quad M \quad K B \quad b b \quad B M ; \quad M / \quad Q \quad \# \quad b \quad 2 \quad \quad p \quad 2 / \quad T \quad \quad i B + B T \quad M i b \quad U \quad B \quad X \quad 2 \quad X \quad J \quad \quad V - \quad r \quad ? \quad B + \quad ? \quad i \quad Q \quad B ; \quad M \quad Q \quad \quad B \quad M ; \quad i \quad ? \quad 2 \quad K B \quad b b \quad B M ; \quad / \quad i \quad X \quad \quad b \quad b \quad m + \quad ? - \quad i \quad ? \quad 2 \quad 2 \quad b \quad i \quad B \quad K \quad i \quad 2 \quad b \quad b \quad ? \quad Q \quad m \quad H / \quad \# \quad 2 \quad p \quad 2 \quad Q \quad \# \quad i \quad B \quad M \quad 2 / \quad m \quad b \quad B \quad M ; \quad i \quad ? \quad 2 \quad M \quad [\quad p \quad 2 \quad X \quad Q \quad / \quad 2 \quad H \quad U \quad h \quad \# \quad H \quad 2$

$A \quad i \quad b \quad ? \quad Q \quad m \quad H / \quad H \quad b \quad Q \quad \# \quad 2 \quad M \quad Q \quad i \quad 2 / \quad i ? \quad i \quad 2 \quad p \quad 2 \quad M \quad i \quad M \beta_{vm} \quad i \quad 4 \quad 2 \quad Q \quad j \quad 2 \quad H \quad v \quad ? \quad 2 \quad Q \quad r \quad 2 \quad i \quad H \quad K \quad 2 \quad M \quad Q \quad 7 \quad 2 \quad z \quad 2 \quad + \quad i \quad i \quad R \quad 3 \quad K \quad Q \quad M \quad B \quad b \quad i \quad B \quad H \quad H \quad M \quad Q \quad M \quad @ \quad b \quad B ; \quad M \quad B \quad \} + \quad M \quad i - \beta_{vm} \quad / \quad B \quad B \quad Q \quad M \quad H \quad 0 \quad K \quad i \quad Q \quad B \quad i \quad ? \quad i \quad b \quad B ; \quad M \quad B \quad \} + \quad M \quad i \quad X \quad Q \quad 2 \quad 2 \quad 2 \quad v \quad b \quad b \quad B ; \quad M \quad B \quad \} + \quad M \quad i \quad ? \quad \quad Q \quad m ; \quad ? \quad Q \quad m \quad i \quad p \quad ? \quad i \quad m \quad 2 \quad b ; \quad 2 \quad Q \quad 7 \quad 2 \quad b \quad 2 \quad Q \quad \# \quad b \quad 2 \quad \quad p \quad i \quad B \quad Q \quad M \quad b \quad T \quad \quad Q \quad p \quad B / \quad 2 \quad ; \quad Q \quad Q / \quad 2 \quad p \quad B / \quad 2 \quad M + \quad 2 \quad i ? \quad i \quad i \quad ? \quad 2 \quad i \quad \quad 2 \quad \quad i \quad K \quad 2 \quad M \quad i \quad 2 \quad z \quad 2 \quad + \quad i \quad B \quad b \quad ? \quad + \quad ? \quad M ; \quad 2 \quad b \quad B \quad M \quad i \quad ? \quad 2 \quad \quad \# \quad B \quad \beta_{vm} \quad X \quad K \quad 2 \quad i \quad 2 \quad \quad$

4.4.2 Bayesian Sensitivity Analysis Where β_{vm} is a Random Variable with Prior Distribution

$q \quad 2 \quad M \quad Q \quad r \quad T \quad \quad 2 \quad b \quad 2 \quad M \quad i \quad i \quad ? \quad 2 \quad \quad " \quad a \quad \quad \quad 2 \quad b \quad m \quad H \quad i \quad b - \quad r \quad ? \quad 2 \quad \quad 2 \quad \quad B \quad M \quad r \quad 2 \quad \quad b \quad b \quad B ; \quad M \quad \quad T \quad \quad B \quad Q \quad \quad T \quad \quad Q \quad \# \quad \# \quad \beta_{vm} \quad X \quad h \quad ? \quad 2 \quad \quad 2 \quad b \quad m \quad H \quad i \quad b \quad 7 \quad \quad Q \quad K \quad a \quad i \quad M \quad + \quad Q \quad K \quad T \quad m \quad i \quad i \quad B \quad Q \quad M \quad b \quad \beta_{vm} \quad b \quad B \quad M ; \quad b \quad m \quad M \quad B \quad 7 \quad Q \quad X \quad 2 \quad / \quad T \quad B \quad M \quad Q \quad \quad 7 \quad Q \quad h \quad \# \quad H \quad 2 \quad X \quad h \quad ? \quad 2 \quad F \quad 2 \quad v \quad Q \quad \# \quad b \quad 2 \quad \quad p \quad i \quad B \quad Q \quad M \quad B \quad b \quad i \quad ? \quad i \quad i \quad ? \quad 2 \quad T \quad Q \quad b \quad i \quad 2 \quad \quad B \quad Q \quad \quad K \quad 2 \quad \quad M \quad b \quad \quad M / \quad N \quad 8 \quad W \quad > \quad S \quad Q \quad 7 \quad \quad 2 \quad \quad \tilde{\beta} \quad \tilde{\beta} \quad \quad M / \quad 2 \quad \quad 2 \quad \quad p \quad 2 \quad \quad v \quad \quad b \quad B \quad K \quad B \quad H \quad \quad i \quad Q \quad i \quad ? \quad 2 \quad B \quad \quad + \quad Q \quad m \quad M \quad i \quad 2 \quad \quad T \quad \quad i \quad b \quad B \quad M \quad i \quad ? \quad 2 \quad M \quad [\quad p \quad 2 \quad " \quad X \quad Q \quad X \quad A \quad M \quad Q \quad i \quad ? \quad 2 \quad \quad r \quad Q \quad / \quad b - \quad i \quad ? \quad 2 \quad \quad " \quad a \quad \quad K \quad Q \quad / \quad 2 \quad H \quad ; \quad B \quad p \quad 2 \quad b \quad p \quad 2 \quad \quad v \quad \quad b \quad B \quad K \quad B \quad H \quad \quad B \quad M \quad 7 \quad 2 \quad \quad 2 \quad M + \quad 2$

h?Bb i2HHb mb i? i i?2 M HvbBb`2bmHib BM i?2 o > / i b2i `2`Q#mb
bbmKtiBQMb #Qmi #B b 7`QK MQM@B;MQ` #H2 KBbbBM; / i X

h #H"2X9HbQ b?Qr_vi??ib TQbi2`BQ` K2 M Q7 @9Xj M/ N8W >S. +
Bmi2`p H Q7 U@kNX3- kjXRVX h?2 +QMbB/2` #H2 rB/i? Q7 i?Bb +`2/
2M+QKT bb2b i?2 KQ`2 2ti_v2K2 p Hm2b Q7

* ? T i 2 ` 8

a B K m H i B Q M b

8 X R : 2 M 2 ` i B M ; a B K m H i 2 / . i b 2 i b

h ? 2 K B M + Q K T ` B b Q M i ? i B b B K T Q ` i M i " B M 2 ? B b b ? 2 # H 2 B b h # H 2
_ 2 b m H i b B M X B # M Q ` 2 b i ? 2 T ` Q # H 2 K Q 7 K B b b B M ; X e i Q - 2 b ? 2 " a b h # H 2

? b 2[m H T`Q# #BHBiv Q7 #2BM; bbB;M2/iQ i?2 i`2 iK2Mi;`QmT U>
Uh IV- b Bb i?2 + b2 BM i?2 +im H` M/QKBx iBQM T`Q+2/m`2X

*Q``2bTQM/BM; iQ 2 +? 2H2K2Mi p MbiB2M2Ki#Q`` - r?B+? ` M;2b 7`Q
iQ 9 UrBi? y b i?2 # b2HBM2V r b +`2 i2/ M XbhQ B2/ iB M m?2 p 2QiQ`
+QMbi`m+i /2bB;M K i`Bt Q7 ymKKv p `B #H2b-

am#b2[m2MiHv- r2 + H+mH i2/ i?2 p2` ;2 H2M;i? Q7 i?2 Ryy +`2/B#
+Qp2` ;2 ` i2 Q7 ~~β~~ iHm2b URy- Ry- Ry- RyVX h?2 +Qp2` ;2 ` i2 Bb /
T`QTQ`iBQM Q7 iBK2b i? i i?2 +`2/B#H2 ~~B~~ M~~p~~2H~~m~~ ~~H~~X+QMi BM i?2 i`m2

Chapter 6

Discussion

AM i?Bb i?2bBb- r2 T`QTQb2/ MQp2H " v2bB M a2MbBiBpBiv M H
b2MbBiBpBiv iQ MQMB;MQ` #H2 KBbbBM;/ i 7Q` i?2 Qmi+QK2 p`B #
KQ/B}2/ T ii2`M KBtim`2 KQ/2H 7Q` i?2 +QKTH2i2 / i BM+Hm/BM; Q#
BM7Q`K iBQM i? i `2 BM/2t2/ #v MQM@B/2MiB} #H2 b2MbBiBpBiv T
i?2 2z2+i Q7 KBbbBM;M2bb QM i?2 Q#b2`p iBQMbX hQ mb2 i?2 K2i?Q/
/Bz2`2Mi p Hm2b Q7 i?2 b2M[~]BiBrpBiv?T+`MK#22BMi2`T`2i2/ b i?2 p2`
/Bz2`2M+2 BM i?2 K2 M ZQG b+Q`2b 7Q` KBbbBM; p2`bmb Q#b2`p2/`
+QKTQM[~]Mi B7 2[m HiQ x2`Q- i?2 i?2 KBbbBM;M2bb Bb B;MQ` #H2 U
M2; iBp2 p Hm2 bm+? b @ky K2 Mb i? i i?2 KBbbBM;M2BpB/m Hb
b+Q`2bX Hi2`M iBp2Hv- T`BQ` T`Q# #BHBiv /Bbi`B#miBQM /2T2M
#2HB27b #Qmi #B b b?Q[~]H/# 2M7b;M2BpQ T`BQ`b + M #2 +QMbi`m+
HBi2` im`2`2pB2rb M/ 2t2`i QTBMbQM i?2 bm#D2+i K ii2`- r?B

>Qr2p2`- 7Q` i?2 "a KQ/2H + `27mH b2H2+iBQM Q7 T`BQ` QM i?2
Bb`2[mB`2/ iQ +QM/m+i K2 MBM;7mH M HvbBb- b BM TT`QT`
BM +QM}/2M+2 BMi2`p Hb Q7 2ti`2K2 rB/i?b Q7 i?2 i`2 iK2Mi 2z2+iX
B/2MiB} #BHBiv Q7 i?2 KQ/2H /B+i i2bi? ii?2 i`m2 p Hm2b Q7 Bib mM/
#2 i?2Q`2iB+ HHv + H+mH i2/ M/ i?2 / i mbm HHv `2p2 H p2`v HBii
Q X *QMb2[m2MiHv- i?2 M Hvb i Kmbi + `27mHHv +?QQb2 i?2 T`BQ
K v ;`2 iHv BM~m2M+2 i?2 M HvbBb `2bmHibX h?Bb +QmH/ +QMbiBi
Bb BMbm{+B2Mi BM7Q`K iBQM iQ /2i2`KBM2 bmBi #H2 +?QB+2bX 6Q`
i?2`2 MQ BM7Q`K iBQM BM i?2 HBi2` im`2 i? i?2H Tbxmb 2biBK i2 i?2

*m`BQmbHv- MQM@BM Q`KQ iBQ 2tT`BQ`#27+Qmb2 i?2 KQ/2H Bb MQM@
aQ 7Q` 2t KTH2- B7 r2 bbB;M0¹⁰π0BQ`iQ 7-IMP27`2bmHiBM; TQbi2`BQ` Q
i?2 i`2 iK2Mi 2z2+ib rQmH/ #2 BM}MBi2Hv rB/2X

MQi?2` /Bb /p Mi ;2 Q7 i?2 "a Bbi? i- b `2 K Mv " v2bB M K2i?Q/
JQM i2 * `HQ bBKmH iBQM b- Bi +QmH/ #2 +QKTmi iBQM HHv BMi2Mb
MmK#2` Q7 KBbbBM; Q#b2`p iBQM b 2tBbi M/ M22/ iQ #2 2biBK i2/X A
Bb +QM bB/2` #H2 +QKTmi iBQM H /B{+mHiv BM " v2bB M J*J* bBKmH
KQ/2H) -(b `2bmHi Q7 +Q`2H i2/ T ` K2i2`b BM M/ B`2;mH `b? T
/Bbi`B#miBQ MX00Xm`2

q2 TTHB2/ i?2 "a K2i?Q/ iQ i?2 o > / i iQ 2biBK i2 i?2 2z2+i Q7 i?2
iBQM QM ZQG Q7 i?2 ?QK2H2bb T `iB+BT MibX AM T `iB+mH ` - r2 mb2
Q7 i?2 M HvbBb `2bmHib iQ /Bz2`2Mi bbmKTiBQM b #Qmi MQM@B;M
72`2Mi p Hm2bXQq2 7QmM/ i? i i?2`2 Bb`2K `F #Hv HBiiH2 b2MbBiBpE
#Qmi KBbbBM; / i X AM Qi?2` rQ`/b- KBbbBM;M2bb Bb MQi ;`2 i+QM
QM i?2 M HvbBb Qmi+QK2b- r?B+? Bb /Qm#iH2bbHv ;Q`8M2rb iQ i?
b?Qrb i? i i?2`2 Bb bHB;?i /Bz2`2M+2 BM Qp2` HH +Q`M+HmbBQM Q
TT`Q +?2b Yjy Q` @jy- +Q`2bTQM/BM; iQ i?2 2ti`2K2 bbmKTiBQM i
KBbbBM; T `iB+BT Mib `2 jy TQBMib ?B;?2` UQ` HQR2`V i? M Qi?2`rE
iB+BT Mib- r?B+? Bb MQi`2 bQM #H2X h?2 ` i?2` HQR H2p2H Q7 b2Mb
7 +i i? i i?2 KBbbBM;M2bb QMHv Ry iQ R8 T2`+2MiX

S2`? Tb i?2 KQbi +m`BQmb `2bmHi Q7 i?2 o > bim/v Bb i?2 /BKBMn
>QmbBM; 6B`bi BM i?2 H i2` bi ;2b Q7 i?2 bim/v- r?B+? r b bQ TT`2M

6.1 Limitations

i?2 2z2+ib Q7 Mv KBbbBM; / i - r?BH2 r2 bBKTHv B;MQ`2/ i?2b2 KB
BKTH2K2Mi iBQM Q7 i?2 M [p2 K2i?Q/X JQ`2Qp2`- i?2 M HvbBb Q7 S
7Q` b2p2` H T`2/B+iQ`b Q7 ZQG- bm+? b ;2 M/ ;2M/2`X am+? +Qp
MQi 2tBbi BM Qm` M HvbBbX hQ ;Bp2 7 B`2` +QKT `BbQM Q7 "a p
HQM;Bim/BM H M HvbBb- BirQmH/ #2 mb27mHiQ //BiBQM HHv TTH

6.2 Future Work

h?2 bBKmH iBQM bim/v r b bQK2r? i HBKBi2/ #2+ m² r2 ;2M2` i2
r b QMHv }t2/ i @8X #2ii2` Hi2`M iBp2 +QmH/ #2 ó" v2bB Mô bBKm
;2M2` i2/ #v` M/QKHv b KTHBM;QKHrQK2Q7` K2i2` b KTHBM;/Bbi`B#n
rBi? i?2 /p Mi ;2 Q7 b?QrBM; ó p2` ;2ô +Qp2` ;2` i2 M/ +QKT `BbC
K2i?Q/ Qp2` M 2MiB`2 /Bbi`B#mpBQM2bX;2Q2`i?2H2bb- i?2 +?QB+
bmBi #H2 T` K2i2` ;2M2` iBM;/Bbi`B#miBQM rQmH/ #2 MQi?2` +QM-

h?2 "a KQ/2HT`QTQb2/ BM i?Bbi?2bBb Bb ` i?2` # bB+ QM2- b Bi
BKTQ`i Mi p `B #H2b- bT2+B}+ HHv iBK2 Q7 pBbBi- i`2 iK2Mi 2z2+i-
M/ ` M/QK 2z2+iX 6m`i?2` KQ/B}+ iBQMb +QmH/ #2 K /2 iQ i?2 "a K
KQ`2 ++m` i2 M HvbBb `2bmHib- bm+? b i?2 BM+Q`TQ` iBQM Q7 i?
K2 bm`2K2Mib ` i?2` i? M Dmbi e KQMi?b BMi2`p H- b r2HH b Qi?2`
HBF2 ;2 M/ ;2M/2`X

Bibliography

(R) >X CX / ` M/ :X CX J2HhB2NB#M2`QM `2b2 `+? K2i?Q/b, +QMbm
+QKT MKBQM? MM2b p M E2bb2H Sm#HBb?BM;X- kyy3X

(k) * M /B M P#b2`p iQ`v QM*> MK/BH2b.2MM BbXQM Q7X-QK-2H@bbM2bb
MB+ H`2TQ`i- * M /B M P#b2`p iQ`v QM >QK2H2bbM2bb- kyRkX

- (kd) .X "X _m#BMX AM72`2M+2B Q K Z B B B B M 8 M ki - X N de X
- (k3) .X "X _m#BM M/ _X CX GBiiH2X ai iBbiB+ HQ #QHf2MB-bLiCB,iC KBbb
qBH2v aQkMpkX
- (kN) CX GX a+? 72` M/ CX qX :` ? KX JBbbBM; / i , Qm` pB2r Q7 i?
Sbv+?QHq;B+-H U 2 M?, Q/9d- kyykX
- (jy) JX JX a? `B iB- AX EQ`b; `/- M/ .X aQ`2Mb2MX A/2MiB} #BHBiv
? pBQm` Q7 K+K+ +? Bmb, + b2 bim/v mbBM; C?Qm`2M+HBQM MQ`I
MBK H "`22/BM; M/ R2M2iBv, Nk Ryk- kyyNX
- (jR) CX JX aQK2`b- JX GX S ii2`bQM- X JQMB`mxx K M- GX *m``B2- aX
M/ EX 6`v2`X o M+Qmp2` i?QK2, T` ;K iB+ ` M/QKBx2/ i`B Hb BM
}`bi7Q` ?QK2H2bb M/ K2Mh` B H B B U R V, jre8 i bLXQ p kyRjX
- (jk) CX X ai2`M2- AX _X q?Bi2- CX "X * `HBM- JX aT` ii- SX _QvbiQM-
qQQ/- M/ CX _X * `T2Mi2`X JmHiBTH2 BKTmi iBQM 7Q` KBbbBM; /
M/ +HBMB+ H`2b2 `+?, TQi2` B i B b? J2// B B i H j C Q # K M j H kyyNX
- (jj) .X hQK bFQpB+@.2p2v- CX G2Bi2`- M/ aX h?QKTbQMX P`; MBx iB
/KBMBbi` iBp2 a+B2M9jNZ98di-2R N W 9 X
- (j9) X "X h`Qt2H- :X J - M/ .X 6X >2BiD MX M BM/2t Q7 HQ+ H b2MbBi
ai iBbiB+ -aBMBR+RkjD- kyy9X
- (j8) :X o2`#2F2- :X JQH2M#2`;?b- >X h?BD b- 1X G2b z`2- M/ JX :X E
M HvbBb 7Q` MQM` M/QK /`QTQmi, HQB-QHK BiMB-4b2RMV,2i R 9`-Q +?)
kyyRX
- (je) .X JX w #FB2rB+x- JX S ii2`bQM- CX 6` MFBb?- M/ CX JX aQK2`
?QK2 bim/v, Pp2`pB2r M/ K2i?Q/b Q7 ?QmbBM;}`bii`B H KQM; B
?QK2H2bb M/ HBpBM; rBiC Q 2 M M H B Q H * M B M B K 9 V H R k B- F b i kyRkX

Appendix A

Code

A.1 Bayesian Linear Mixed Effects Model that Ignores Missing Data

```
## Stan model specification

Ine_code <- '

data {
  int n;          // number of participants
  int nk;        // number of records
  int id[nk];    // participant id
  real y[nk];    //  $y$ 
  real x[nk];    // group assignment indicator
  matrix[nk, 4] v;
  // design matrix of dummy variables to indicate visit time
}

parameters {
  real theta[n]; // random effects
  real mu_theta; // mean of the random effects
  vector[4] betav; // time effect
  vector[4] betax; // treatment effect

  real <lower=0> sigma_theta; // sd of the random effects
  real <lower=0> sigma; // sd of the  $y$ 
}

model {
  mu_theta ~ normal(0, 10); // priors
  betav ~ normal(0, 10);
  betax ~ normal(0, 10);
  sigma_theta ~ normal(0, 10);
  sigma ~ normal(0, 10);
  theta ~ normal(mu_theta, sigma_theta);
}
```

```

for (i in 1:nk)
  y[i] ~ normal(mu[i] * betav + (mu[i] * betax) * x[i] + theta[id[i]], sigma);
// model @
}

# Translate Stan model specification to C++ code
tcode <- stanc(model_code = line_code, model_name = `mymodel`, verbose = TRUE)

# Construct Stan model
tmodel <- stan_model(stanc_ret = tcode, verbose = FALSE)

# Specify list of data
ah.data <- list(id, y, x, v, n, nk)

# Sample from Stan model
tsamples <- sampling(tmodel, data = ah.data, chains = 1, iter = 200)
thin = 1, verbose = TRUE)

# Extract MCMC samples
MCMC.samples <- extract(tsamples)

```

A.2 Bayesian Sensitivity Analysis for Non-ignorable Missing Data

```

# Stan model specification
bsa_code <- '

data {
  int n; // number of participants
  int ndos // number of observed records
  int nmis // number of missing records
  int id[ndos+nmis]; // participant id
  real yos[ndos]; // observed @

  real x[ndos+nmis]; // group assignment indicator
  int m[ndos+nmis]; // missing data indicator
  matrix x[ndos+nmis, 4] v
  // design matrix of dummy variables to indicate visit time
}

parameters {
  real theta[n]; // random effects
  real mu_theta; // mean of the random effects
  vector[4] betav // time effect
  vector[4] betax // treatment effect

  real <lower=0> sigma_theta; // sd of the random effects
  real <lower=0> sigma; // sd of the @
  real ym[nmis]; // missing @

```

```

real gamm0 // logistic regression parameter
real gammx // logistic regression parameter

real <lower= 0, upper=3> betavn // bias due to miss ingness
}

model {
  mu_theta ~ normal(0, 10); // pri ors
  betav ~ normal(0, 10);
  betavx ~ normal(0, 10);
  sigma_theta ~ normal(0, 10000);
  sigma ~ normal(0, 10000);
  theta ~ normal(mu_theta, sigma_theta);

  gamm0 ~ normal(0, 10);
  gammx ~ normal(0, 10);

  for (i in 1:(nmiss))
    ymiss[i] ~ normal((v[i+ndos] * (betav + betavn) + (v[i+ndos] * betavx) *
    x[i+ndos] + theta[id[i+ndos]]), sigma);
    // imputing the missing data using the true value of betavn

  for (i in 1:(ndos))
    yobs[i] ~ normal((v[i] * betav + (v[i] * betavx) * x[i] +
    theta[id[i]]), sigma);
    // analysis of the observed data

  for (i in 1:(nmiss+ndos))
    m[i] ~ bernoulli(logit(gamm0 + gammx * x[i]));
    // logistic regression for estimating the proportion missing parameters
}

## Translate Stan model specification to C++ code
tcode <- stanc(model_code = bsa_code, model_name = `mymodel`, verbose = TRUE)

## Construct Stan model
tmodel <- stan_model(stanc_ret = tcode, verbose = FALSE)

## Specify list of data
ah.data <- list(id, yobs, x, v, m, n, ndos, nmiss)

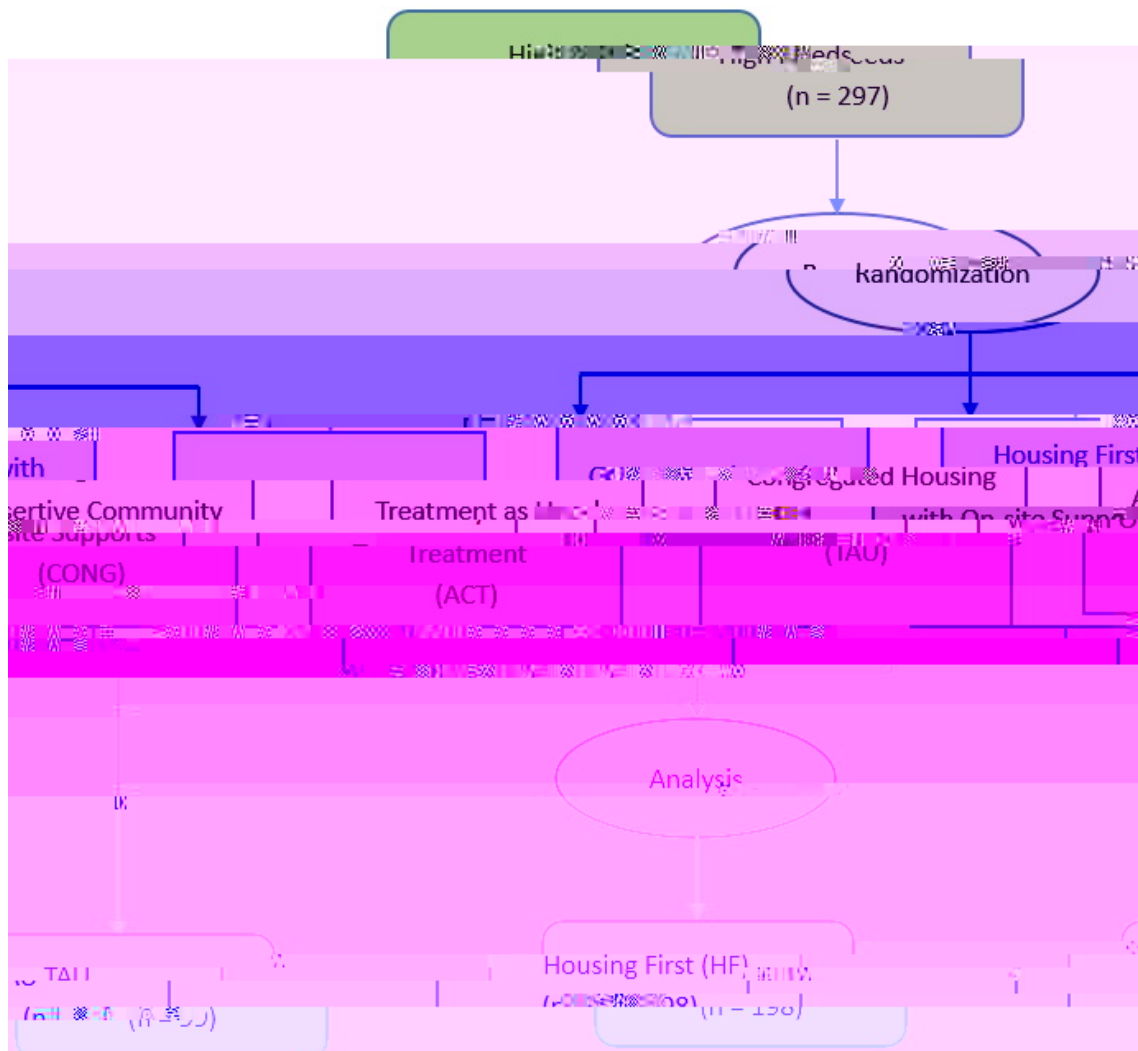
## Sample from Stan model
tsamples <- sampling(tmodel, data = ah.data, chains = 1, iter = 200)
thin = 1, verbose = TRUE)

## Extract MCMC samples
MCMC.samples <- extract(tsamples)

```


Appendix B

Tables and Figures



6B;m`2 "XR, :`QmT HHQ+ iBQM Q7 ?B;? M22/b T `iB+BT Mib BM i

o`B #H2	*QmMi US2`+2Mi ;2bV	J2/B M UAZ_V
>QmbBM; 6B`bi Up2`bmbRnI3/UeeXdWV @		
J H2	kRk UdkX9WV @	
;2		
	Ik8 k9 U3XkWV @	
	k8@9y RdN UeR XR @	
	=9y Ny UjyXdWV @	
O >2 Hi? *QM/BiBQMb @		9 Uk@dV
ZQG @		3d Udy@RykV
*aA @		jy UkR@9RV

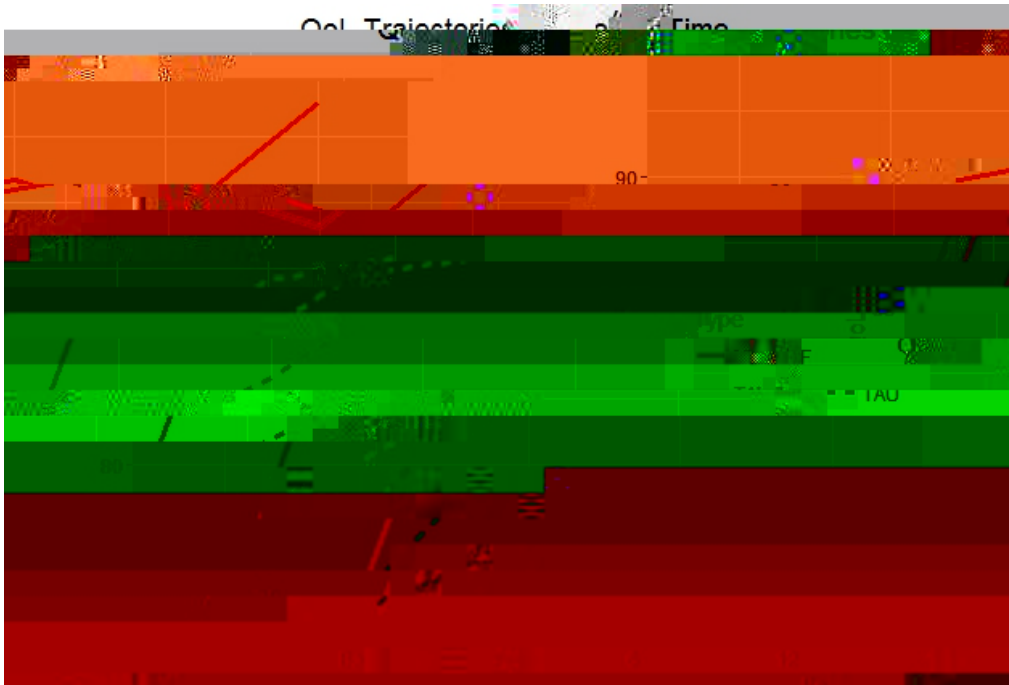
h #H2 "XR, .2b+`BTiBp2 bi iBbiB+b Q7 i?2 o M+Qmp2` i >QK2 / i b2
BM/BpB/m Hb rBi? K2Mi H BHHM2bb

oBbBi	" b2HBM2 e KQMi?b	Rk KQMi?b	R3 KQMi?b	k9 K
LmK#2` Q7 T `iB+BT MibdykNd ke9		k9d	kjR	

h #H2 "Xk, LmK#2` Q7 T `iB+BT Mib i# b2HBM2 M/ 2 +?`2p

oBbBi hBK2	>6 J2 Mh l J2 M± a.	T@p Hm2
" b2HBM2 ±dlj R jke	d9 *R R 8	yX33RN
e JQMi?b 3±NkKdRj	3y±B B Xd	yXyyk3
Rk JQMi?b N y X Xj	3j ±j R N X 8	yXyRRe
R3 JQMi?b 3 N X X R	3± R y N X k	yX8jy3
k9 JQMi?b N j k X Xj	3± X B X 3	yXy3y3

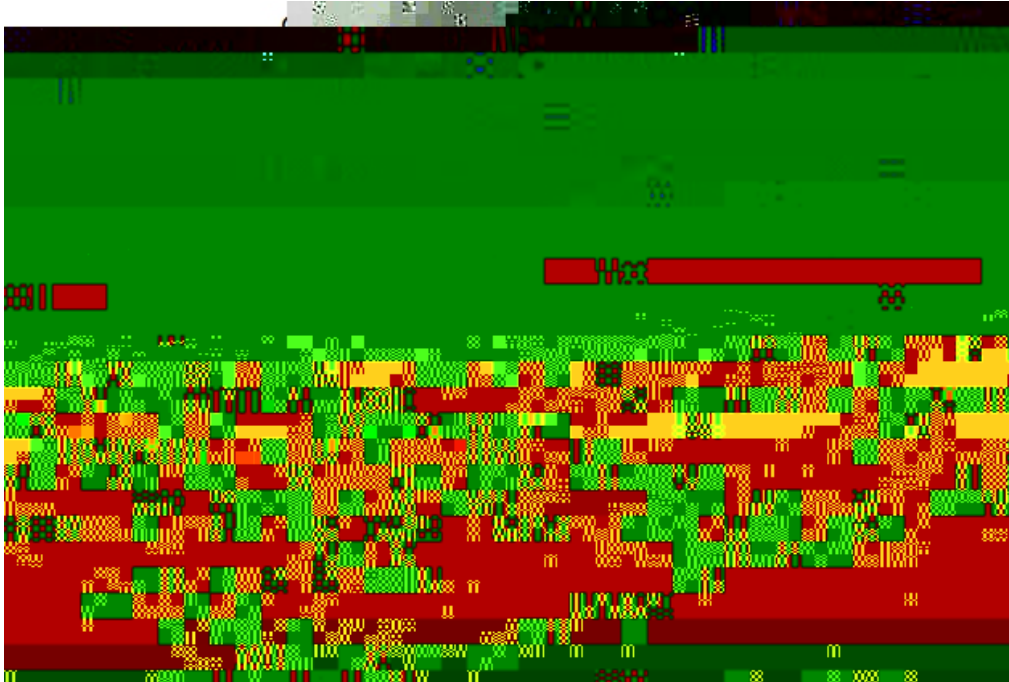
h #H2 "Xj, ZQG K2 M b+Q`2b i# b2HBM2 M/ e- Rk- R3 M/ k9 KQMi?
`K



6B;m`2 "Xk, p2` ;2 ZQG i` D2+iQ`B2b BM >6 M/ h l ;`QmTb



6B;m`2 "Xj, AM/BpB/m H ZQG i` D2+iQ`B2b 7Q` ` M/QK b KTH2 Q7 b E
>6 ;`QmT



6B;m`2 "X9, AM/BpB/m H ZQG i` D2+iQ`B2b 7Q` ` M/QK b KTH2 Q7 b
h l ;`QmT

o `B #H2	SQbi2`BQ` J2 M N8W >S. *A	
hBK2		
" b2HBM2		
e JQM _{v1} b	eX8	UkXe- RRXNV
Rk JQM _{v2} b	RyXj	U8XN- R8XRV
R3 JQM _{v3} b	R9Xy	UNX9- R3XNV
k9 JQM _{v4} b	R8Xy	URyX9- RNX3V
hBK2 >QmbBM; 6B`bi		
>6x e JQM _{v1} b	3X9	UkXN- RjXdV
>6x Rk JQM _{v2} b	8XN	UyXd- RyXNV
>6x R3 JQM _{v3} b	yXd	U@9X3- eXyV
>6x k9 JQM _{v4} b	9Xj	U@RX8- NX8V
	d9X8	UdRX3- ddXyV
	R8Xe	URjX3- RdXjV
	ReXe	UR8XN- RdXjV

h #H2 "X9, SQbi2`BQ` K2 M M/ N8W >S. +`2/B#H2 BMi2`p H Q7 p `B
" v2bB M HQM;Bim/BM H KQ/2H i? i B;MQ`2b KBbbBM; / i UM [p2 KQ/

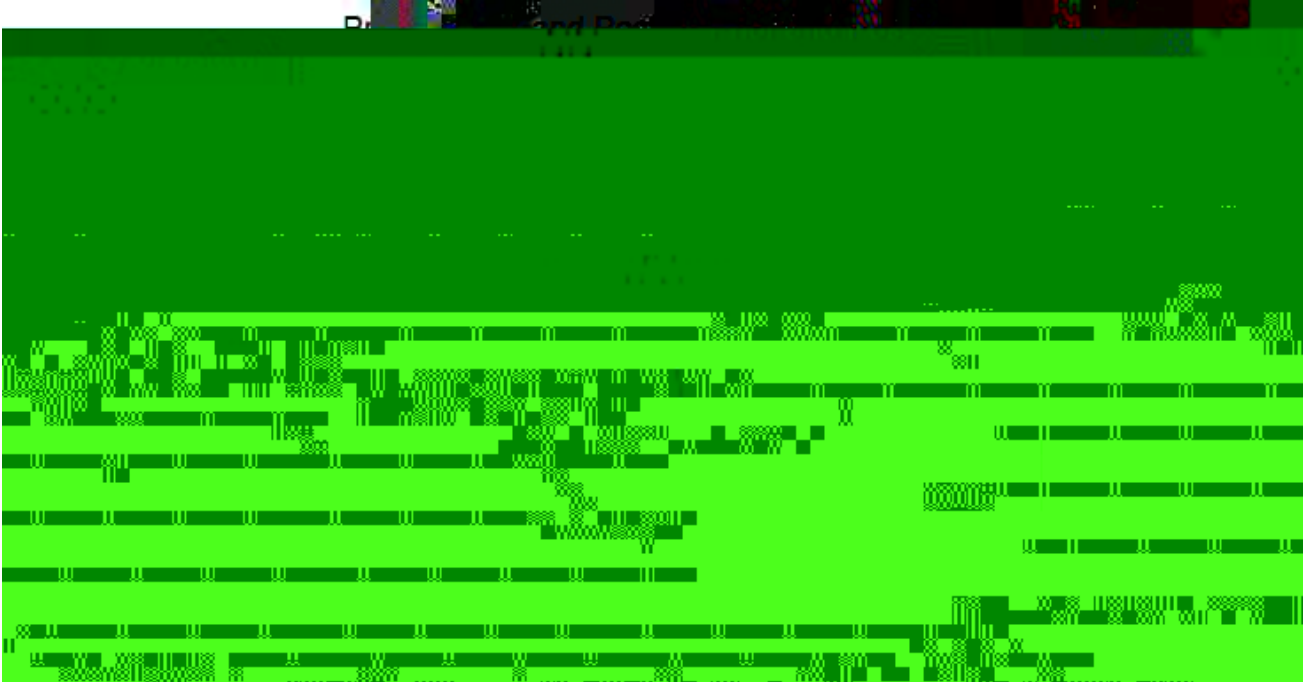
β_{vm}	β_{vx1}	β_{vx2}	β_{vx3}	β_{vx4}
jy	8X8 UyX3-	RRXR V	jXy U@RX3-	3X3V @kXR U@dXj-
ky	eXk UyXN-	RRX3V	jXe U@RXd-	NXdV @RX8 U@dXR-
Ry	dXj UkXR-	RkXkV	9XN U@yX8-	NXNV @yXj U@8Xd-
y	3X9 UjX9-	RjXNV	8X3 UyX3-	RRX9V yXe U@9Xk-
@Ry	NXk U9X9-	R9XdV	eXN URX3-	RRX3V RX3 U@jXj-
@ky	RyXj U8Xj-	R8XkV	dXe UkXe-	RkXNV kX9 U@kXN-
@jy	RRXR U8Xd-	R8XNV	3Xd UjXk-	RjXNV jX8 U@kXk-

h #H2 "X8, SQbi2`BQ` K2 M M/ N8W >S. +`2/B#H2 Bmi2`p H Q7 Qp2`
 β_{vm} 4 & @jy- @ky- @Ry- y- Ry- ky- jM/xBi? p `B2/

o `B #H2	SQbi2`BQ` J2 M	N8W >S. *A
hBK2	" b2HBM2	
	e JQM β_{v1} b	eXN UkX8- RRXyV
	Rk JQM β_{v2} b	RyXe UeXk- R9XNV
	R3 JQM β_{v3} b	R9Xk UNX8- R3XeV
	k9 JQM β_{v4} b	R8X9 URyXe- kyX3V
hBK2 > QmbBM; 6B`bi	>6x e JQM β_{v1} b	3X9 UkXj- RjX3V
	>6x Rk JQM β_{v2} b	eXR UyX3- RkXyV
	>6x R3 JQM β_{v3} b	RXy U@9X3- dXeV
	>6x k9 JQM β_{v4} b	9Xk U@kX8- RyX8V
		d9X9 UdRXN- ddXyV
		R8X8 URjXN- RdXR V
		ReXe UR8XN- RdXjV
β_{vm}	@9Xj	U@kNX3- kjXR V

h #H2 "Xe, SQbi2`BQ` K2 M M/ N8W >S. +`2/B#H2 Bmi2`p H Q7 KQ/2H
b2MbBiBpBiv M HvbBb U"a V TTHB2/ iQ i?2 i >QK2 / i mbBM; m M
/Bbi`B#miBQM QM i?2 #B b T `K2i2`

J2i?Q	β_{vx1}	β_{vx2}	β_{vx3}	β_{vx4}
	QpX G2	M;QpX G2	M*;QpX G2	M;i?



6B;m`2 "X8, S`BQ` M/ TQb_m BQ`i /B_m2/B_mv_m B_m; " v2bB M b2MbBiBpBiv