Faculty of Science
Department of Mathematics

MATH 875 PhD Preliminary Examination - SAMPLE

Candidate Identification

Last name:

First name:

Student ID:

SFUID:

Question	Full marks	Marks Obtained
	10	
	10	
	10	
	10	
	10	
	10	
TOTAL	60	

Exam Instructions

The duration of the exam is three hours.

The examination booklet consists of three pages, including this one. There are ten questions.

Solutions to a total of 6 questions will be considered towards a final mark.

Please write your answers in the enclosed booklet, and return this document with your answer booklet.

Each question will be graded out of 10. To obtain full marks a solution must be well-written, mathematically correct, and complete. The solution should use results and arguments that a third year student would be reasonably expected to know.

Questions

Complete only six of the following ten questions.

1. (a) Find all feasible critical points of

$$\min f(x) = x_1^2 - 2x_1 + x_2^2 - x_3^2 + 4x_3; \quad \text{subject to} \quad x_1 - x_2 + 2x_3 = 2.$$

   (b) Determine a basis for the null-space corresponding to the constraint normal, and use it to verify the second order optimality condition. Classify the critical points as local max, min, or saddle points.

   (c) Restate the problem from part (a) as a quadratic program in the from

$$\min \frac{1}{2} x^T Q x + c^T x + \qquad \text{subject to} \quad Ax = b;$$

   for suitable matrices $(A; Q)$, vectors $(b; c)$, and scalar .

2. Consider the differential equation

$$x^2 y'' + 5xy' + 4y = 0$$

   with $x > 0$ and initial conditions $y(1) = 2$, $y'(1) = 3$.

   (a) Solve the above initial value problem.

   (b) Describe how the solution behaves as $x \to 0$.

3. Let $V = C([-1; 1])$ be the vector space of continuous real-valued functions on the interval $[-1; 1]$ with inner product $\langle f; g \rangle = \int_{-1}^{1} f(t)g(t)\,dt$

(b) Show that if $f(z) = \log(z)(r > 0, \quad < \quad < \quad + 2\ )$, then f is analytic in this region, and find its derivative.

(c) If $z = \sin w$, find an expression for $\sin^{\ 1} z$ in terms of the logarithm. Discuss how to make this single-valued.

9. Let D be an orientation of a tree with the property that $\deg\ (v) \quad 1$ for every $v \in V(D)$. Prove that there exists a vertex $r \in V(D)$ so that every vertex $v \in V(D)$ can be reached from r by a directed path. (Hint: how many edges are in a tree?)

10. The *Stirling number of the second kind* $S(n,k)$ is defined as the number of ways to partition the set $\{1, 2, \ldots, n\}$ into k blocks. Prove that $S(n,k)$ satisfies

$$S(n+1, k) = \sum_{j=0} \binom{n}{j} S(n \quad j, k \quad 1).$$